Models of Computation

4: Regular expressions, finite automaton

Regular expressions

Applications

- search and replace dialogs of text editors
- search engines
- text processing utilities (e.g. sed and AWK)
- programming languages, lexical analysis
- genom analysis (genom as string)
- spam/malware filter
- ...

Regular expressions

Let *V* and *V'* = {ε, ·, +, * , (,)} be disjoint alphabets. A **regular expression** over V is defined recursively as follows:

1. ε is a regular expression over *V*,
2. all a ∈ *V* are regular expressions over *V*,
3. If *R* is a regular expression over *V*, then *R** is also a regular expression over *V*,
4. If *Q* and *R* are regular expressions over *V*, then (*Q* · *R*) and (*Q* + *R*) are also regular expressions over *V*.

* denotes the closure of iteration,
· the concatenation, and
\+ union.

Regular expressions

Each regular expression **represents a regular language**, which is defined as:

1. ε represents the language {ε},
2. Letter *a* (∈ *V*) represents the language {*a*},
3. if *R* is a regular expression over *V*, which represents the language *L*, then *R** represents *L**,
4. if *Q* and *R* are regular expressions over *V*, that represent the languages *L* and *L'*, then
   (*Q* · *R*) represents the language *LL'*,
   (*Q* + *R*) represents the language *L* U *L'*.

Regular expressions

Parentheses can be omitted when defining precedence on operations. The the usual sequence is: *, ·, +. The following regular expressions are equivalent:

- a^* is the same as $(a)^*$ and represent the language $\{a\}^*$.
- $(a + b)^*$ is the same as $((a) + (b))^*$ and represents the language $\{a, b\}^*$.
- $a^* \cdot$ b is the same as $((a)^*) \cdot (b)$ and represents the language $\{a\}^*b$.
- $b + ab^*$ is the same as $(b) + ((a) \cdot (b)^*)$ and represents the language $\{b\} \cup \{a\}\{b\}^*$.
- $(a + b) \cdot a^*$ is the same as $((a) + (b)) \cdot ((a)^*)$ and represents the language $\{a, b\}\{a\}^*$.

Regular expressions

Let P, Q, an R be regular expressions. Then following hold:

- $P + (Q + R) = (P + Q) + R$
- $P \cdot (Q \cdot R) = (P \cdot Q) \cdot R$
- $P + Q = Q + P$
- $P \cdot (Q + R) = P \cdot Q + P \cdot R$
- $(P + Q) \cdot R = P \cdot R + Q \cdot R$
- $P^* = \varepsilon + P \cdot P^*$
- $\varepsilon \cdot P = P \cdot \varepsilon = P$
- $P^* = (\varepsilon + P)^*$

Regular expressions

Example:

The language represented the regular expressions
$(a + b)a^*$ and $aa^* + ba^*$ is the same:
$\{ aa^n \mid n \in \mathbb{N}\} \cup \{ ba^n \mid n \in \mathbb{N} \}$.

The language represented by $a + ba^*$ is:
$\{ a, b, ba, ba^2, ba^3, \ldots\}$.

Expressive power of regular expressions

Theorem:

1) Every regular expression represents a regular (3-type) language.

2) For every regular (3-type) language, there is a regular expression representing the language.

Proof:

1) follows from the fact that the class of regular languages $\mathcal{L}_3$ is closed for the regular operations.

Expressive power of regular expressions

Proof:

For 2), we show that for every regular language L generate by a grammar $G = (N, T, P, S)$, a regular expression can be constructed, that represents L.

- Let $N = \{A_1, . . . , A_n\}$, $n \geq 1$, $S = A_1$.
  - Each rule of G is of form $A_i \rightarrow aA_j$ or $A_i \rightarrow \varepsilon$, where $a \in T$, $1 \leq i, j \leq n$.
- We say that a non-terminal A_m is **affected** by the derivation
  $A_i \Rightarrow^* uA_j$ ($u \in T^*$), if A_m occurs in a intermediate string between A_i and uA_m in the derivation.

Expressive power of regular expressions

Proof (cont.):

- A derivation $A_i \Rightarrow^* uA_j$ is called **k-bounded** if $0 \leq m \leq k$ holds for all non-terminals A_m occurring in the derivation.
- Let $E^k_{i,j} = \{u \in T^* \mid \exists\, A_i \Rightarrow^* uA_j$ k-bounded derivation$\}$.
- We show by induction on k, that for language $E^k_{i,j}$, there is a regular expression representing $E^k_{i,j}$, where $0 \leq i,j,k \leq n$.

Expressive power of regular expressions

Proof (cont.):

- k=0 (induction start):
  - For $i \neq j$, $E^{0}_{i,j}$ is eighter empty, or it consists of symbols of T ($a \in E^{0}_{i,j}$ if and only if $A_i \rightarrow aA_j \in P$.)
  - For $i = j$, $E^{0}_{i,j}$ consists of ε and zero or more elements of T, so $E^{0}_{i,j}$ can be represented by a regular expression.

Expressive power of regular expressions

Proof (cont.):

- k-1 –>k (induction step):
  - Assume that for fixed k, $0 < k \leq n$, $E^{k-1}_{i,j}$ can be represented by a regular expression.
  - Then for all i, j, k it holds that
    - $E^{k}_{i,j} = E^{k-1}_{i,j} + E^{k-1}_{i,k} \cdot (E^{k-1}_{k,k})^* \cdot E^{k-1}_{k,j}$.
  - Therefore, $E^{k}_{i,j}$ can also be represented by a regular expression.
- Let I_ε be the set of indices i for which $A_i \rightarrow \varepsilon$.
  - Then $L(G) = \bigcup_{i \in I_\varepsilon} E^{n}_{1,i}$ can be representd by a regular expression. The claim of the theorem follows.

Finite Automata (FA)

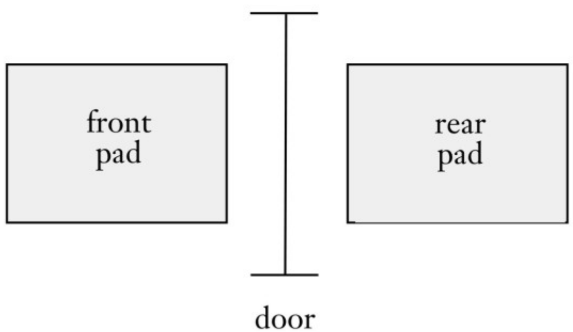
- Identifying formal languages is also possible with recognition devices, i.e. by automata.
- An automaton can process and identify words.
- Grammars use a synthesizing approach, while automata an analytic one.
- In response to a word, the automaton can either accept or reject.

Finite Automata (FA)

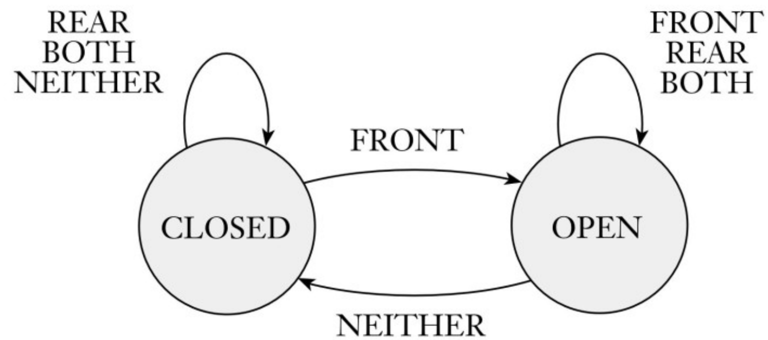
- A finite automaton performs a sequence of steps in discrete time intervals
- It starts in the initial state.
- The input word is located on the input tape and the reading head is on the leftmost symbol of an input word.
- After reading a symbol, the automaton moves the reading head to one position to the right, then the state changes, regarding the state transition function.
- If the automaton has read the input, it stops (accepts or rejects the input).

Finite Automata (FA)

- Example: automatic door control



State transition diagram:



State transition table:

		input signal			
		NEITHER	FRONT	REAR	BOTH
state	CLOSED	CLOSED	OPEN	CLOSED	CLOSED
	OPEN	CLOSED	OPEN	OPEN	OPEN

Finite Automata (FA)

- Application examples:
  - Automatic door control
  - Coffee machine
  - Pattern recognition
  - Markov chains – pattern recognition
  - Speech processing
  - Optical character recognition
  - Predictions of share prizes in the stock exchange
  - ...

Finite Automata (FA)

A finite automaton is a 5-tuple $A = (Q, T, \delta, q_0, F)$, where

- Q is a finite, nonempty set of **states**,
- T is the finite **alphabet of input symbols**,
- $\delta : Q \times T \to Q$ is the **state transition function**
- $q_0 \in Q$ is the **initial state** or **start state**,
- $F \subseteq Q$ is the set of **acceptance states** or **end states**.

Finite Automata (FA)

Remark:

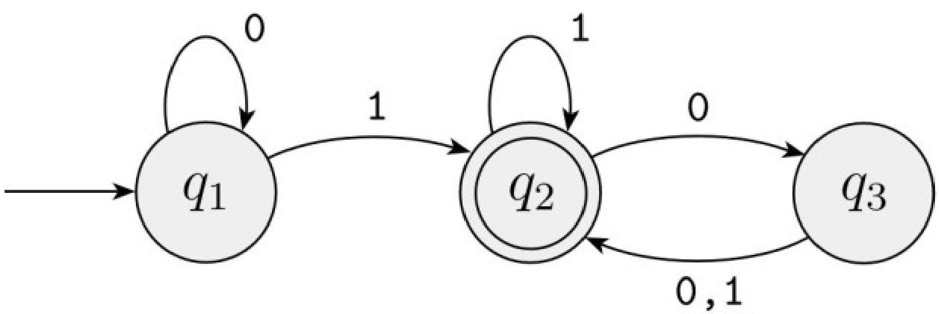
- The function δ can be extended to a function $\hat{\delta} : Q \times T^* \to Q$ as follows:
  - $\hat{\delta}(q, \varepsilon) = q$,
  - $\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$ for all $x \in T^*$ and $a \in T$.

Finite Automata (FA)

Example:

- Let $A = (Q, T, \delta, q_1, F)$ be a FA, where
  $Q = \{q_1, q_2, q_3\}$, $T = \{0, 1\}$, $F = \{q_2\}$, and
  $\delta(q_1, 0) = q_1$, $\delta(q_1, 1) = q_2$, $\delta(q_2, 0) = q_3$, $\delta(q_2, 1) = q_2$,
  $\delta(q_3, 0) = \delta(q_3, 1) = q_2$.

State transition diagram:



State transition table:

δ	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

- The accepted language is $L(A)=\{w \mid w$ conains at least one 1 and the last 1 is not followed by an odd number of 0s$\}$

Finite Automata (FA)

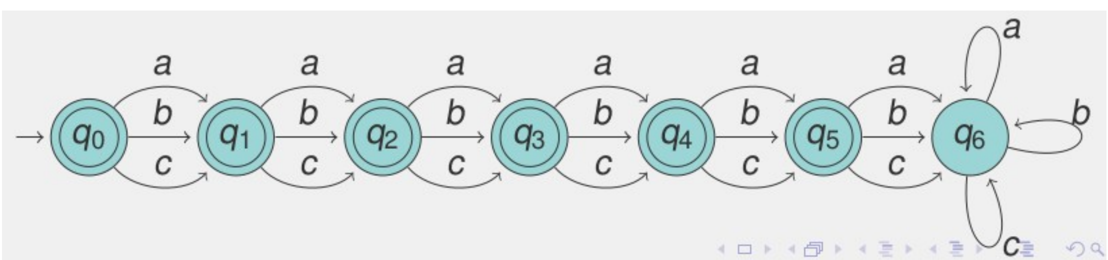
Example:

- Let $T = \{a,b,c\}$.
  Define a FA, which accepts the words of length of at most 5.

Solution:

- Formaly:
  $A=(\{q_0, \ldots, q_6\}, \{a, b, c\}, \delta, q_0, \{q_0, \ldots, q_5\})$,
  $\delta(q_i, t) = q_{i+1}$, for $i = 0, \ldots, 5$, $t \in \{a, b, c\}$,
  $\delta(q_6, t) = q_6$, for $t \in \{a, b, c\}$

- By state transition diagram:



By state transition table:

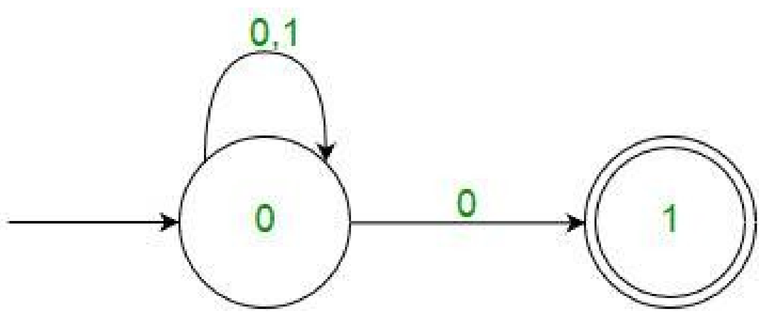
	a	b	c
$\leftrightarrows q_0$	q_1	q_1	q_1
$\leftarrow q_1$	q_2	q_2	q_2
$\leftarrow q_2$	q_3	q_3	q_3
$\leftarrow q_3$	q_4	q_4	q_4
$\leftarrow q_4$	q_5	q_5	q_5
$\leftarrow q_5$	q_6	q_6	q_6
q_6	q_6	q_6	q_6

FA – accepted language

- The **language accepted/recognized** by the FA $A = (Q, T, \delta, Q_0, F)$ is:
  $L(A) = \{u \in T^* \mid q_0 u \Rightarrow^* p$ for some $q_0 \in Q_0$ and $p \in F\}$
- For a deterministic FA A, there is one single start state $Q_0=\{q_0\}$. The language accepted by DFA A is:
  $L(A) = \{u \in T^* \mid q_0 u \Rightarrow^* p$ for some $p \in F\}$

Deterministic and non-deterministic finite automata

- **Deterministic finite automaton (DFA)**: Function δ is single-valued, i.e. $\forall\ (q, a) \in Q \times T$ there is exactly one state s, s.t. $\delta(q, a) = s$.
- **Nondeterministic finite automaton (NFA)**: Function δ is multi-valued, i.e. $\delta : Q \times T \rightarrow 2^Q$.
  Multiple initial states are allowed (the set of initial states $Q_0 \subseteq Q$).
  It is allowed that $\delta(q, a) = \emptyset$ for some (q,a), i.e. the machine gets stuck.
  Null (or ε) move is allowed, i.e. it can move forward without reading symbols.



NFA example

Deterministic and non-deterministic finite automata

- New features of non-determinism
  - Multiple paths are possible (multiple choises at each step).
  - ε-transition is a “free” move without reading input.
  - Accept input if <u>some</u> path leads to an accepting state.

Deterministic and non-deterministic finite automata

- Alternative notation:
- State transitions can also be given in the form $qa \rightarrow p$, where $p \in \delta(q, a)$.
- Let M_δ be set of rules of the state transition of an NFA $A = (Q, T, \delta, Q_0, F)$.
- If M_δ contains exactly one rule $qa \rightarrow p$ for each pair (q,a), then the FA is deterministic, oherwise non-deterministic.

FA – reduction

- Let $A = (Q, T, \delta, Q_0, F)$ be a FA and $u,v \in QT^*$ words. The FA A **reduces** the u **in one step** (**directly**) to v (notation: $u \Rightarrow_A v$, or short: $u \Rightarrow v$), if there are a rule $qa \to p \in M_\delta$ (i.e. $\delta(q, a) = p$) and a word $w \in T^*$, s.t. $u = qaw$ and $v = pw$ hold.
- The FA $A = (Q, T, \delta, Q_0, F)$ **reduces** $u \in QT^*$ to $v \in QT^*$ (notation: $u \Rightarrow_A^* v$, or short: $u \Rightarrow^* v$, if
  - either $u = v$,
  - or $\exists$ a word $z \in QT^*$, s.t. $u \Rightarrow^* z$ and $z \Rightarrow$ v.
- Remark: $\Rightarrow^*$ is the reflexive, transitive closure of $\Rightarrow$.

FA – accepted language

- The **language accepted/recognized** by the FA $A = (Q, T, \delta, Q_0, F)$ is:
  $L(A) = \{u \in T^* \mid q_0u \Rightarrow^* p$ for some $q_0 \in Q_0$ and $p \in F\}$
- For a DFA A, there is one single start state $Q_0=\{q_0\}$. The language accepted by DFA A is:
  $L(A) = \{u \in T^* \mid q_0u \Rightarrow^* p$ for some $p \in F\}$

Computing power of non-deterministic FA

- **Theorem**: For all NFA $A = (Q, T, \delta, Q_0, F)$ a DFA $A' = (Q', T, \delta', q'_0, F')$ can be constructed, s.t. $L(A) = L(A')$ holds.

- Idea: DFA keeps track of the subset of possible states in NFA
- Remark: In worst case $|Q'| = 2^{|Q|}$.

Computing power of non-deterministic FA

Proof:

- Let $Q' = 2^Q$ be the set of all subsets of the set Q. (the number of elements of Q' is $2^{|Q|}$).
- Let $\delta' : Q' \times T \rightarrow Q'$ be the function defined as: $\delta'(q', a) = \bigcup_{q \in q'} \delta(q, a)$.
- Let $q'_0 = Q_0$ and $F' = \{q' \in Q' \mid q' \cap F \neq \emptyset\}$
- To prove $L(A) \subseteq L(A')$, we prove the Lemma 1:

Computing power of non-deterministic FA

Lemma 1:

- For all $p,q \in Q$, $q' \in Q'$ és $u,v \in T^*$,
  if $qu \Rightarrow^*_A pv$ and $q \in q'$,
  then $\exists\ p' \in Q'$, s.t.
  $q'u \Rightarrow^*_{A'} p'v$ and $p \in p'$.

Proof:

- Induction over the number of reduction steps n in $qu \Rightarrow^*_A pv$.
- For $n=0$: the claim holds trivially, $p'=q'$.

Computing power of non-deterministic FA

Proof (Lemma 1, cont.):

- For $n \rightarrow n+1$: Assume, the claim holds for all reductions of $\leq n$ steps.
- Let $qu \Rightarrow^*_A pv$ be a reduction of $n + 1$ steps.
  Then for some $q_1 \in Q$ and $u_1 \in T^*$ holds that
  $qu \Rightarrow_A q_1u_1 \Rightarrow^*_A pv$.
- Therefore, $\exists\ a \in T$, s.t. $u = au_1$ and $q_1 \in \delta(q, a)$.
- Since $\delta(q, a) \subseteq \delta'(q', a)$, for $q \in q'$,
  q'_1 can be choosen as $q'_1 = \delta'(q', a)$.
- Consequently, $q'u \Rightarrow_{A'} q'_1u_1$, where $q_1 \in q'_1$.
- By the induction assumption,
  $\exists\ p' \in Q'$, s.t. $q'_1u_1 \Rightarrow^*_{A'} p'v$ and $p \in p'$, which proves the claim. □

Computing power of non-deterministic FA

Proof (Theorem, cont.):

- Let $u \in L(A)$, i.e. $q_0 u \Rightarrow^*_A p$, for some $q_0 \in Q_0$, $p \in F$.
- By Lemma 1, $\exists\, p'$, s.t. $q'_0 u \Rightarrow^*_{A'} p'$, where $p \in p'$.
- By definition of F', $p \in p'$ and $p \in F$ imply that $p' \in F'$, which proves $L(A) \subseteq L(A')$.
- For $L(A') \subseteq L(A)$ we prove Lemma 2.

Computing power of non-deterministic FA

Lemma 2:

- For all $p', q' \in Q'$, $p \in Q$ and $u, v \in T^*$,
  - if $q'u \Rightarrow^*_{A'} p'v$ and $p \in p'$,
  - then $\exists\, q \in Q$, s.t. $qu \Rightarrow^*_A pv$ and $q \in q'$.

Proof:

- Induction over the number of steps *n* in the reduction.
- For n = 0: The claim holds trivially.

Computing power of non-deterministic FA

Proof (Lemma 2, cont.):

- For $n \to n+1$: Assume, the claim holds for all reductions of $\leq n$ steps.
- Let $q'u \Rightarrow^*_{A'} p'v$ be a reduction of $n + 1$ steps.
  Then $q'u \Rightarrow^*_{A'} p'_1 v_1 \Rightarrow_{A'} p'v$, where $v_1 = av$,
  for some $p'_1 \in Q'$ and $a \in T$.
- Then, $p \in p' = \delta'(p'_1, a) = \bigcup_{p_1 \in p'_1} \delta(p_1, a)$.
- Consequently, $\exists\, p_1 \in p'_1$, s.t. $p \in \delta(p_1, a)$.
- For this p_1, it holds that $p_1 v_1 \Rightarrow_A pv$.
- By the induction assumption, $qu \Rightarrow^*_A p_1 v_1$, for some $q \in q_0$, which implies the claim. □

Computing power of non-deterministic FA

Proof (Theorem, cont.):

- Let $q'_0 u \Rightarrow^*_{A'} p'$ and $p' \in F$.
- By the definition of F', $\exists\, p \in p'$, s.t. $p \in F$.
- Then, by Lemma 2, for some $q_0 \in q'_0$, holds that $q_0 u \Rightarrow^*_A p$.
- This proves the claim of the theorem. □

NFA – DFA

Example:

- Let $A = (Q,T,\delta,Q_0,F)$ be a NFA, where
  $Q = \{q_0, q_1, q_2\}$, $T = \{a, b\}$, $Q_0 = \{q_0\}$, $F = \{q_2\}$.
  δ is defined as:
  $\delta(q_0, a) = \{q_0, q_1\}$, $\delta(q_0, b) = \{q_1\}$,
  $\delta(q_1, a) = \emptyset$, $\delta(q_1, b) = \{q_2\}$,
  $\delta(q_2, a) = \{q_0, q_1, q_2\}$, $\delta(q_2, b) = \{q_1\}$.
  Construct a DFA A' quivalent with A.

Solution:

- DFA: $A' = (Q',T,\delta',q'_0,F')$, where
  $Q' = \{\emptyset,\{q_0\},\{q_1\},\{q_2\},\{q_0, q_1\},\{q_0,q_2\},\{q_1,q_2\},\{q_0,q_1,q_2\}\}$,
  $q'_0 = \{q_0\}$,
  $F' = \{\{q_2\},\{q_0,q_2\},\{q_1,q_2\},\{q_0,q_1,q_2\}\}$,
  next slide

NFA – DFA

Example (cont.):

- δ:

$$\begin{array}{ll} \delta(q_0, a) = \{q_0, q_1\}, & \delta(q_0, b) = \{q_1\}, \\ \delta(q_1, a) = \emptyset, & \delta(q_1, b) = \{q_2\}, \\ \delta(q_2, a) = \{q_0, q_1, q_2\}, & \delta(q_2, b) = \{q_1\}. \end{array}$$

- δ':

$$\begin{array}{cc} \delta'((\emptyset, a) = \emptyset, & \delta'((\emptyset, b) = \emptyset, \\ \delta'((\{q_0\}, a) = \{q_0, q_1\}, & \delta'((\{q_0\}, b) = \{q_1\}, \\ \delta'((\{q_1\}, a) = \emptyset, & \delta'((\{q_1\}, b) = \{q_2\}, \\ \delta'((\{q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_2\}, b) = \{q_1\}, \\ \delta'((\{q_0, q_1\}, a) = \{q_0, q_1\}, & \delta'((\{q_0, q_1\}, b) = \{q_1, q_2\}, \\ \delta'((\{q_0, q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_0, q_2\}, b) = \{q_1\}, \\ \delta'((\{q_1, q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_1, q_2\}, b) = \{q_1, q_2\}, \\ \delta'((\{q_0, q_1, q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_0, q_1, q_2\}, b) = \{q_1, q_2\}. \end{array}$$

Corollaries

Corollary 1:

- The class of regular languages $\mathcal{L}_3$ is closed for the complement operation.

Proof:

- Let L be a language, recognized by a FA $A = (Q,T,\delta,q_0,F)$
- Then $\overline{L} = T^* - L$ can be recognized by an FA $A = (Q,T,\delta,q_0,Q-F)$

Corollaries

Corollary 2:

- The class of regular languages $\mathcal{L}_3$ is closed for the intersection operation.

Proof:

- We know, that $\mathcal{L}_3$ is closed for the union operation.
- $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$.
- By Corollary 1, the claim follows.

Corollaries

Corollary 3:

- It is decidable, whether two regular grammars generate the same language

Proof:

- Let G_1 and G_2 be regular grammars, generating the languages L_1 and L_2, respectively
- The language $L_3 = (L_1 \cap \overline{L}_2) \cup (\overline{L}_1 \cap L_2)$ is also regular.
- Consequently, there exists a regular grammar G_3, which generates L_3.
- However, $L_1 = L_2$ holds if and only if $L_3 = \emptyset$, which is decidable for all context-free grammars G_3 (Thus, for all regular grammars).

FA – Myhill-Nerode Theorem

- Let L be a language over the alphabet T. The **relation E_L induced by language L** is a binary relation on T^*, for which it holds that
  $\forall\, u, v \in T^*$, uE_Lv, if and only if $\nexists\, w \in T^*$, s.t. exatly one of the words uw and vw is an element of L (i.e. $\forall\, w \in T^* : uw \in L$ if and only if $vw \in L$).
- E_L is an **equivalence relation** and it is **right-invariant**. (Right-invariant: if uE_Lv, then uwE_Lvw holds for every word $w \in T^*$.)
- The **index of the E_L** is the number of its equivalence classes.

Theorem (Myhill-Nerode): $L \subseteq T^*$ can be recognized by a deterministic FA if and only if E_L has a finite index.

FA – Myhill-Nerode Theorem

Theorem (Myhill-Nerode): $L \subseteq T^*$ can be recognized by a DFA if and only if E_L has a finite index.

- This index is equal to the number of states in the minimal DFA recognizing L.

DFA with minimum number of states

- The DFA *A* has a minimum number of states (**minimal DFA**), if there is no DFA *A'*, which recognizes the same language as *A*, but the number of states of *A'* is smaller than the number of state of *A*.

Theorem: The minimal DFA accepting the regular language *L* is unique, up to isomorphism.

DFA with minimum number of states

Theorem: The minimal DFA accepting the regular language L is unique, up to isomorphism.

- Let $A = (Q, T, \delta, q_0, F)$ be a DFA. Define a relation $R \subseteq Q \times Q$, s.t. pRq if $\forall$ input word $x \in T^*$ it holds that $px \Rightarrow^*_A r$ if and only if $qx \Rightarrow^*_A r'$ for some $r, r' \in F$ states. ($r = r'$ is possible).
- States p and q are **distinguishable** if $\exists\, x \in T^*$, s.t. either $px \Rightarrow^*_A r$, $r \in F$, or $qx \Rightarrow^*_A r'$, $r' \in F$, but both reductions are not possible. Otherwise, p and q are **indistinguishable**.
- If p and q are indistinguishable, then $\delta(p, a) = s$ and $\delta(q, a) = t$ are indistinguishable for any $a \in T$.
- If $\delta(p, a) = s$ and $\delta(q, a) = t$ are distinguishable for $x \in T^*$, then they are distinguishable also for ax.

DFA with minimum number of states

- Let $A = (Q, T, \delta, q_0, F)$ be a DFA. State q is **reachable** from the initial state if there is a reduction $q_0x \Rightarrow^* q$, where x is some word over T.
- The DFA $A = (Q, T, \delta, q_0, F)$ is **connected**, if all its states are reachable from the initial state.
- We define the **set *H* of reachabele states** as follows:
  Let $H_0 = \{q_0\}$, $H_{i+1} = H_i \cup \{r \mid \delta(q, a) = r , q \in H_i , a \in T\}$, $i = 1, 2, ...$
  Then $\exists$ k ≥ 0 : $H_k = H_l$, for all $l \geq k$. Let $H = H_k$.
- We define the DFA $A' = (Q', T, \delta', q_0, F')$ with
  $Q' = H$, $F' = F \cap H$ and $\delta' : H \times T \rightarrow H$ s.t. $\delta'(q, a) = \delta(q, a)$,
  if $q \in H$.
- It can be shown that A' is connected and accepts the same language as A. Furthermore, A' is the **largest connected subautomaton** of A.

DFA with minimum number of states

- In order to compute the minimal DFA,
  - we determine whether the automaton is connected or not.
    - if it is not connected, then we make it connected and consider the largest connected subautomaton. In the following, we assume, that it is connected.
  - then we partition (according to distinguishability, states become divided into equivalence classes

DFA with minimum number of states

- **Step 1**:
  - Divide the set of states into two partitions: F and $Q - F$.
    - The states in F can be distinguished from the states in $Q - F$ (by the empty word).
  - Repeat splitting of the partitions (Step 2) into additional partitions as long as the number of partitions remains the same.
- **Step 2**:
  - This is done as follows: Consider an arbitrary partition P of states. Take an input symbol a and consider $\delta(p, a)$ for each state $p \in P$.
    If the obtained states belong to different partitions, then split P into as many new partitions as arosing in this way.
  - Perform this procedure for each input symbol and each partition, until no new partition is created.

DFA with minimum number of states

- **Step 3**:
  - Determine the DFA with the minimum number of states components.
    - For each partition B_i, consider a representative state b_i.
    - Construct a DFA $A' = (Q', T, \delta', q_0, F')$, where
    - Q' is set of representatives of the partitions,
    - q'_0 is the representative of the partition containing q_0,
    - $\delta'(b_i, a) = b_j$, if $\exists\ q_i \in B_i$ and $q_j \in B_j$, s.t. $\delta(q_i, a) = q_j$.
    - $F' = \{b_f\}$ is the representative of the partition that contains the elements of F.