Models of Computation

4: Regular expressions, finite automaton

Regular expressions

Applications

- search and replace dialogs of text editors
- search engines
- text processing utilities (e.g. sed and AWK)
- programming languages, lexical analysis
- genom analysis (genom as string)
- spam/malware filter
- ...

Regular expressions

Let V and $V' = \{\emptyset, \varepsilon, \cdot, +, {*}, (,)\}$ be disjoint alphabets.
A **regular expression** over V is defined recursively as follows:

1. $\emptyset$ is a regular expression over V,
2. ε is a regular expression over V,
3. a is a regular expression over V, for every $a \in V$,
4. If R is a regular expression over V, then R^* is also a regular expression over V,
5. If Q and R are regular expressions over V, then $(Q \cdot R)$ and $(Q + R)$ are also regular expressions over V.

$*$ denotes the closure of iteration,
$\cdot$ concatenation, and
$+$ union.

Regular expressions

Each regular expression **represents a regular language**, which is defined as:

1. $\emptyset$ represents the empty language,
2. ε represents the language {ε},
3. Letter a ($\in V$) represents the language $\{a\}$,
4. if R is a regular expression over V, which represents the language L, then R^* represents L^*,
5. if R and R' are regular expressions over V, s.t.
   R represents the language L and
   R' represents the language L', then
   $(R \cdot R')$ represents the language LL',
   $(R + R')$ represents the language $L \cup L'$.

Regular expressions

Parentheses can be omitted when defining precedence on operations. The the usual sequence is: *, ·, +. The following regular expressions are equivalent:

- a^* is the same as $(a)^*$ and represent the language $\{a\}^*$.
- $(a + b)^*$ is the same as $((a) + (b))^*$ and represents the language $\{a, b\}^*$.
- $a^* \cdot$ b is the same as $((a)^*) \cdot (b)$ and represents the language $\{a\}^*b$.
- $b + ab^*$ is the same as $(b) + ((a) \cdot (b)^*)$ and represents the language $\{b\} \cup \{a\}\{b\}^*$.
- $(a + b) \cdot a^*$ is the same as $((a) + (b)) \cdot ((a)^*)$ and represents the language $\{a, b\}\{a\}^*$.

Regular expressions

Let P, Q, an R be regular expressions. Then following hold:

- $P + (Q + R) = (P + Q) + R$
- $P \cdot (Q \cdot R) = (P \cdot Q) \cdot R$
- $P + Q = Q + P$
- $P \cdot (Q + R) = P \cdot Q + P \cdot R$
- $(P + Q) \cdot R = P \cdot R + Q \cdot R$
- $P^* = \varepsilon + P \cdot P^*$
- $\varepsilon \cdot P = P \cdot \varepsilon = P$
- $P^* = (\varepsilon + P)^*$

Regular expressions

Example:

The language represented the regular expressions $(a + b)a^*$ and $aa^* + ba^*$ is the same: $\{ aa^n \mid n \geq 0 \} \cup \{ ba^n \mid n \geq 0 \}$.

The language represented by $a + ba^*$ is: $\{ a, b, ba, ba^2, ba^3, \ldots\}$.

Expressive power of regular expressions

Theorem:

1) Every regular expression represents a regular (type 3) language.

2) For every regular (type 3) language, there is a regular expression representing the language.

Proof:
1) follows from the fact that the class of regular languages $\mathcal{L}_3$ is closed for the regular operations.

Expressive power of regular expressions

Proof:

For 2), we show that for every regular language L generated by a grammar $G = (N, T, P, S)$, a regular expression can be constructed, that represents L.

- Let $N = \{A_1, \ldots, A_n\}$, $n \geq 1$, $S = A_1$.
- Assume, each rule of G is of form $A_i \rightarrow aA_j$ or $A_i \rightarrow \varepsilon$, where $a \in T$, $1 \leq i, j \leq n$.
- We say that a non-terminal A_m is **affected** by the derivation $A_i \Rightarrow^* uA_j$ ($u \in T^*$), if A_m occurs in an intermediate string between A_i and uA_j in the derivation.

Expressive power of regular expressions

Proof (cont.):

- A derivation $A_i \Rightarrow^* uA_j$ is called ***k*-bounded** if $0 \leq m \leq k$ holds for all non-terminals A_m occurring in the derivation.
- Let $E^k_{i,j} = \{u \in T^* \mid \exists\, A_i \Rightarrow^* uA_j$ k-bounded derivation$\}$.
- We show by induction on k, that for language $E^k_{i,j}$, there is a regular expression representing $E^k_{i,j}$, $0 \leq i,j,k \leq n$.

Expressive power of regular expressions

Proof (cont.):

- $k = 0$ (induction start):
  - For $i \neq j$, $E^0_{i,j}$ is eighter empty, or it consists of symbols of T ($a \in E^0_{i,j}$ if and only if $A_i \rightarrow aA_j \in P$.)
  - For $i = j$, $E^0_{i,j}$ consists of ε and zero or more elements of T, so $E^0_{i,j}$ can be represented by a regular expression.

Expressive power of regular expressions

Proof (cont.):

- k-1 → k (induction step):
  - Assume that for a fixed k, $0 < k \le n$, $E^{k-1}_{i,j}$ can be represented by a regular expression.
  - Then for all i, j, k it holds that $E^{k}_{i,j} = E^{k-1}_{i,j} + E^{k-1}_{i,k} \cdot (E^{k-1}_{k,k})^* \cdot E^{k-1}_{k,j}$.
  - Therefore, $E^{k}_{i,j}$ can also be represented by a regular expression.
  - Let I_ε be the set of indices i for which $A_i \to \varepsilon$.
  - Then $L(G) = \bigcup_{i \in I_\varepsilon} E^{n}_{1,i}$ can be representd by a regular expression. The claim of the theorem follows. □

Finite Automata (FA)

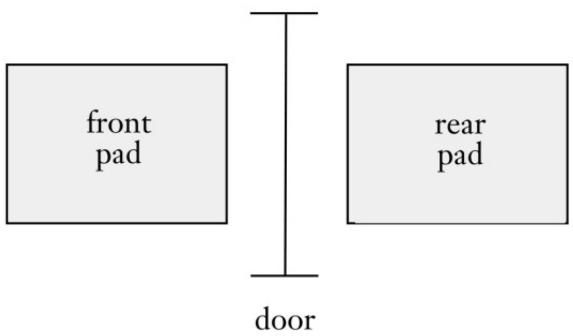
- Identifying formal languages is also possible with recognition devices, i.e. by automata.
- An automaton can process and identify words.
- Grammars use a synthesizing approach, while automata an analytic one.
- An automaton accepts or rejects an input word.

Finite Automata (FA)

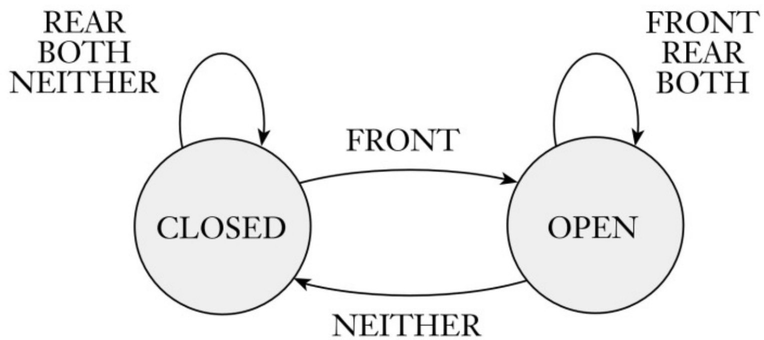
- A finite automaton (FA) performs a sequence of steps in discrete time intervals
- The FA starts in the initial state.
- The input word is located on the input tape and the reading head is on the leftmost symbol of an input word.
- After reading a symbol, the FA moves the reading head to one position to the right, then the state changes, regarding the state transition function.
- If the FA has read the input, it stops, accepts or rejects the input.

Finite Automata (FA)

- Example: automatic door control



State transition diagram:



State transition table:

input signal

state		NEITHER	FRONT	REAR	BOTH
	CLOSED	CLOSED	OPEN	CLOSED	CLOSED
	OPEN	CLOSED	OPEN	OPEN	OPEN

Finite Automata (FA)

- Application examples:
  - Automatic door control
  - Coffee machine
  - Pattern recognition
  - Markov chains
  - Pattern recognition
  - Speech processing
  - Optical character recognition
  - Predictions of share prizes in the stock exchange
  - ...

Finite Automata (FA)

A finite automaton is a 5-tuple $A = (Q, T, \delta, q_0, F)$, where

- Q is a finite, nonempty set of **states**,
- T is the finite **alphabet of input symbols**,
- $\delta : Q \times T \to Q$ is the **state transition function**
- $q_0 \in Q$ is the **initial state** or **start state**,
- $F \subseteq Q$ is the set of **acceptance states** or **end states**.

Finite Automata (FA)

Remark:

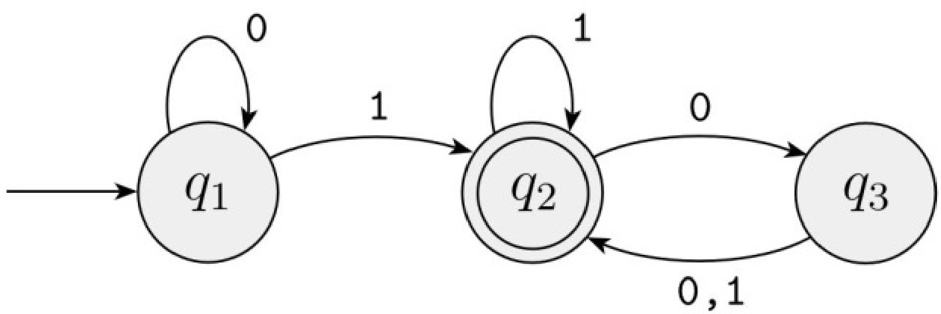
- The function δ can be extended to a function $\hat{\delta} : Q \times T^* \to Q$ as follows:
  - $\hat{\delta}(q, \varepsilon) = q$,
  - $\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$ for all $x \in T^*$ and $a \in T$.

Finite Automata (FA)

Example:

- Let $A = (Q, T, \delta, q_1, F)$ be a FA, where
  $Q = \{q_1, q_2, q_3\}$, $T = \{0, 1\}$, $F = \{q_2\}$, and
  $\delta(q_1, 0) = q_1$, $\delta(q_1, 1) = q_2$, $\delta(q_2, 0) = q_3$, $\delta(q_2, 1) = q_2$,
  $\delta(q_3, 0) = \delta(q_3, 1) = q_2$.

State transition diagram:



State transition table:

δ	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

- The accepted language is $L(A)=\{w \mid w$ conains at least one 1 and the last 1 is not followed by an odd number of 0s$\}$

Finite Automata (FA)

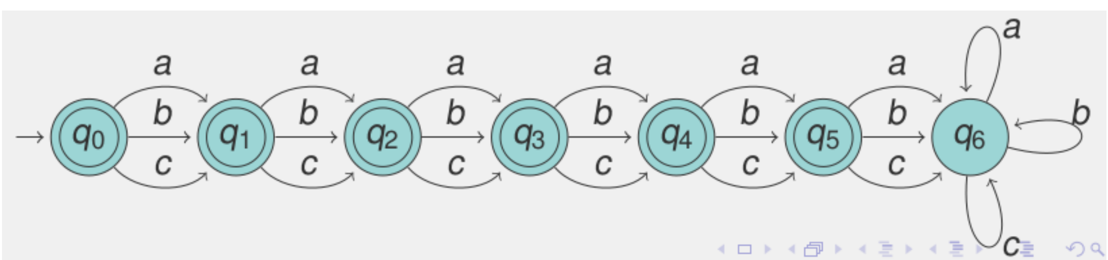
Example:

- Let $T = \{a,b,c\}$.
  Define a FA, which accepts the words of length of at most 5.

Solution:

- Formaly:
  $A=(\{q_0, \ldots, q_6\}, \{a, b, c\}, \delta, q_0, \{q_0, \ldots, q_5\})$,
  $\delta(q_i, t) = q_{i+1}$, for $i = 0, \ldots, 5$, $t \in \{a, b, c\}$,
  $\delta(q_6, t) = q_6$, for $t \in \{a, b, c\}$

- State transition diagram:

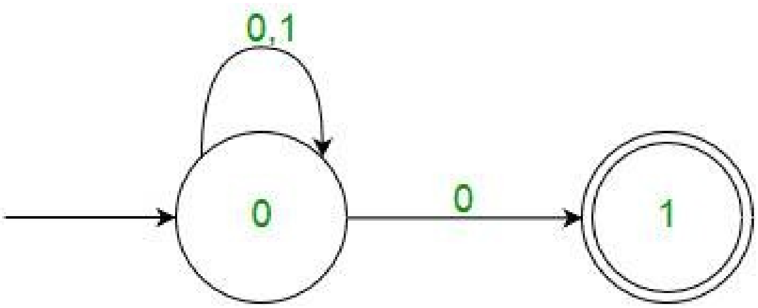


State transition table:

	a	b	c
⇆q_0	q_1	q_1	q_1
←q_1	q_2	q_2	q_2
←q_2	q_3	q_3	q_3
←q_3	q_4	q_4	q_4
←q_4	q_5	q_5	q_5
←q_5	q_6	q_6	q_6
q_6	q_6	q_6	q_6

Deterministic and non-deterministic finite automata

- **Deterministic finite automaton (DFA)**:
  Function δ is single-valued, i.e. $\forall\ (q, a) \in Q \times T$ there is exactly one state s, s.t. $\delta(q, a) = s$.
- **Nondeterministic finite automaton (NFA)**:
  - Function δ is multi-valued, i.e. $\delta : Q \times T \rightarrow 2^Q$.
  - Multiple initial states are allowed (the set of initial states $Q_0 \subseteq Q$).
  - It is allowed that $\delta(q, a) = \emptyset$ for so (q,a), i.e. the machine gets stuck
  - Null (or ε) move is allowed, i.e. it can move forward without reading symbols.



NFA example

Deterministic and non-deterministic FA

- New features of non-determinism
  - Multiple paths are possible (multiple choises at each step).
  - ε-transition is a “free” move without reading input.
  - Accepts the input if <u>some</u> path leads to an accepting state.

Deterministic and non-deterministic FA

- Alternative notation:
- State transitions can also be given in the form $qa \rightarrow p$, where $p \in \delta(q, a)$.
- Let M_δ be set of rules of the state transition of an NFA $A = (Q, T, \delta, Q_0, F)$.
- If M_δ contains exactly one rule $qa \rightarrow p$ for each pair (q,a), then the FA is deterministic, oherwise non-deterministic.

FA – reduction

- Let $A = (Q, T, \delta, Q_0, F)$ be a FA and $u,v \in QT^*$ words.
  The FA A **reduces** the u **in one step** (**directly**) to v (notation: $u \Rightarrow_A v$, or short: $u \Rightarrow v$), if there are a rule $qa \to p \in M_\delta$ (i.e. $\delta(q, a) = p$) and a word $w \in T^*$, s.t. $u = qaw$ and $v = pw$ hold.
- The FA $A = (Q, T, \delta, Q_0, F)$ **reduces** $u \in QT^*$ to $v \in QT^*$ (notation: $u \Rightarrow_A^* v$, or short: $u \Rightarrow^* v$, if
  - either $u = v$,
  - or $\exists$ a word $z \in QT^*$, s.t. $u \Rightarrow^* z$ and $z \Rightarrow v$.

- Remark: $\Rightarrow^*$ is the reflexive, transitive closure of $\Rightarrow$.

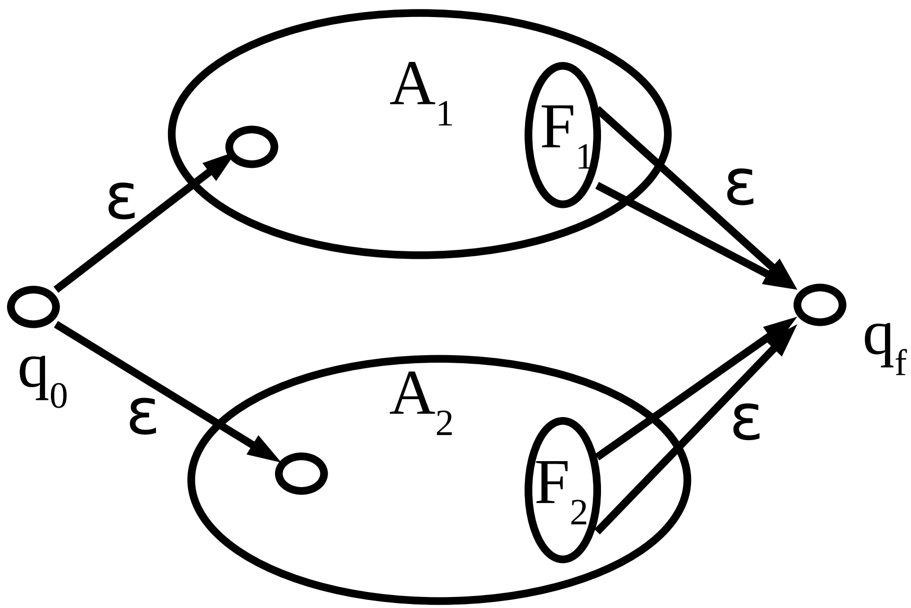
FA – accepted language

- The **language accepted/recognized** by the FA $A = (Q, T, \delta, Q_0, F)$ is:
  $L(A) = \{u \in T^* \mid q_0u \Rightarrow^* p$ for some $q_0 \in Q_0$ and $p \in F\}$
- For a DFA A, there is one single start state $Q_0=\{q_0\}$. The language accepted by DFA A is:
  $L(A) = \{u \in T^* \mid q_0u \Rightarrow^* p$ for some $p \in F\}$

NFA accepting $L_1 \cup L_2$

Theorem: If L_1 and L_2 are regular languages, then $L_1 \cup L_2$ is also a regular language.

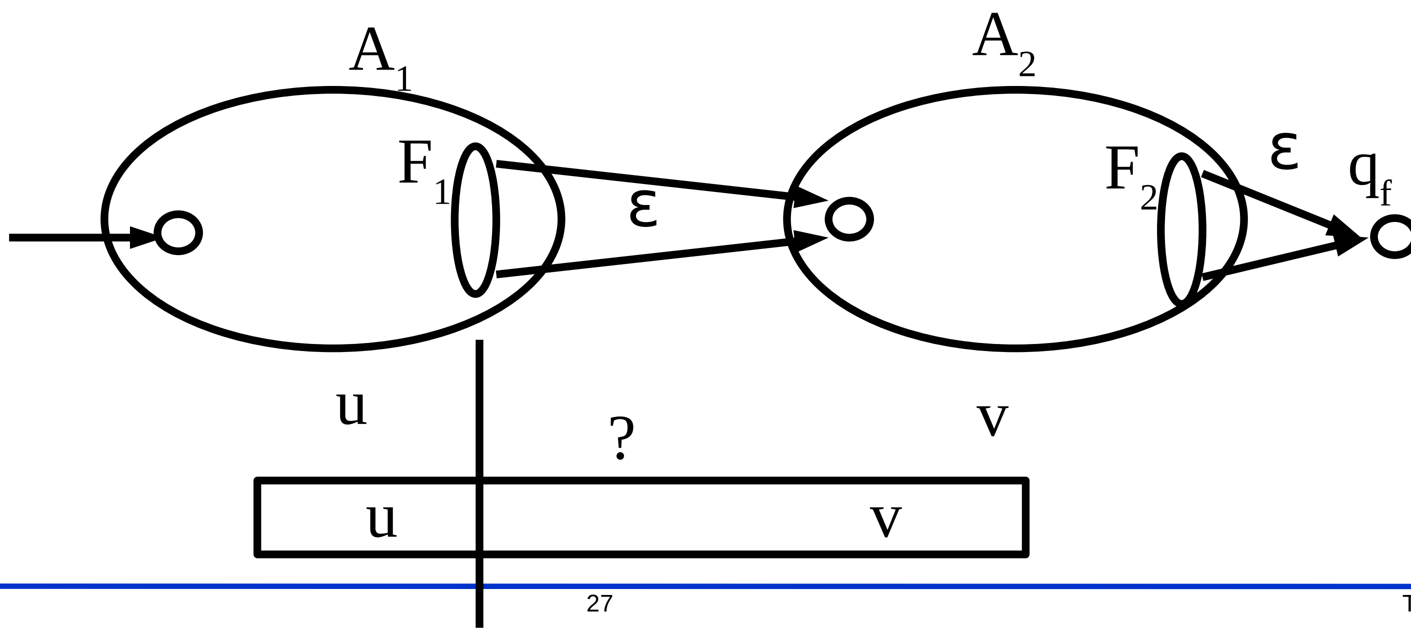
Proof (sketch): Let A_1 be a DFA, accepting L_1 and A_2 a DFA accepting L_2. Then the following NFA accepts $L_1 \cup L_2$.



NFA accepting L_1L_2

Theorem: If L_1 and L_2 regular languages, then L_1L_2 is also a regular language.

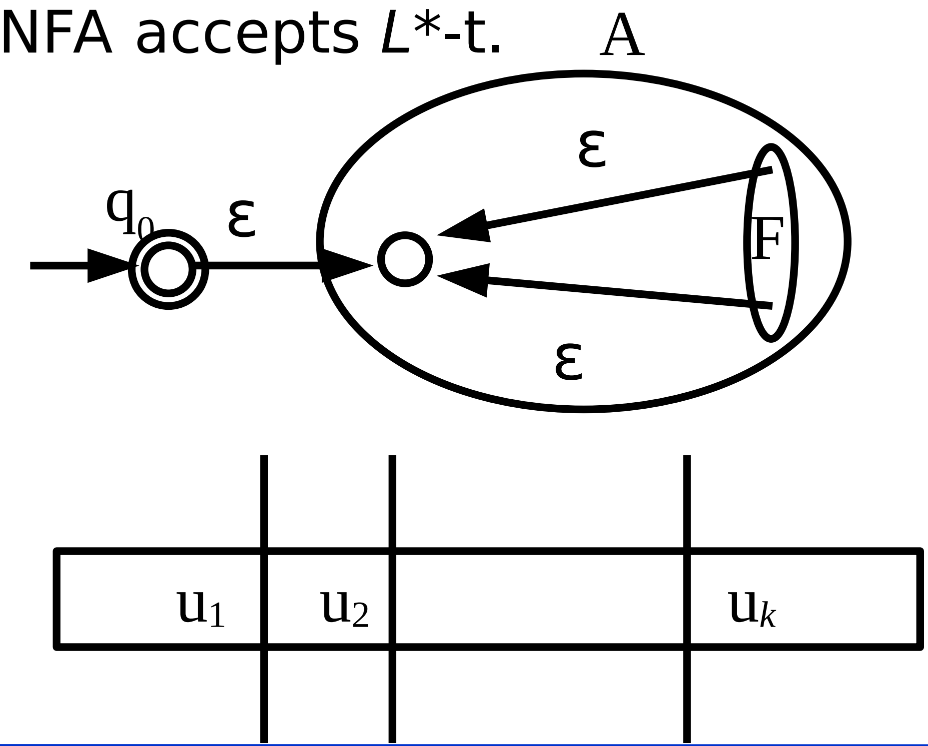
Proof (sketch): Let A_1 be a DFA accepting L_1,
A_2 egy DFA accepting L_2.
The following NVA accepts L_1L_2.



NFA accepting *L**

Theorem: If *L* is a regular language, then *L** is also a regular language.

Proof. (sketch): Let *A* be a DFA accepting *L.*
The fillowing NFA accepts *L**-t.



Computing power of NFA

- **Theorem**: For all NFA $A = (Q, T, \delta, Q_0, F)$ a DFA $A' = (Q', T, \delta', q'_0, F')$ can be constructed, s.t. $L(A) = L(A')$ holds.

- Idea: DFA keeps track of the subset of possible states in NFA
- Remark: In worst case $|Q'| = 2^{|Q|}$.

Computing power of NFA

Proof:

- Let $Q' = 2^Q$ be the set of all subsets of the set Q. (the number of elements of Q' is $2^{|Q|}$).
- Let $\delta' : Q' \times T \rightarrow Q'$ be the function defined as: $\delta'(q', a) = \bigcup_{q \in q'} \delta(q, a)$.
- Let $q'_0 = Q_0$ and $F' = \{q' \in Q' \mid q' \cap F \neq \emptyset\}$
- To prove $L(A) \subseteq L(A')$ we prove Lemma 1, to $L(A') \subseteq L(A)$ we prove Lemma 2.
- First, an example (next slide)

NFA – DFA

Example:

- Let $A = (Q,T,\delta,Q_0,F)$ be a NFA, where
  $Q = \{q_0, q_1, q_2\}$, $T = \{a, b\}$, $Q_0 = \{q_0\}$, $F = \{q_2\}$.
  δ is defined as:
  $\delta(q_0, a) = \{q_0, q_1\}$, $\delta(q_0, b) = \{q_1\}$,
  $\delta(q_1, a) = \emptyset$, $\delta(q_1, b) = \{q_2\}$,
  $\delta(q_2, a) = \{q_0, q_1, q_2\}$, $\delta(q_2, b) = \{q_1\}$.
  Construct a DFA A' quivalent with A.

Solution:

- DFA: $A' = (Q',T,\delta',q'_0,F')$, where
  $Q' = \{\emptyset,\{q_0\},\{q_1\},\{q_2\},\{q_0, q_1\},\{q_0,q_2\},\{q_1,q_2\},\{q_0,q_1,q_2\}\}$,
  $q'_0 = \{q_0\}$,
  $F' = \{\{q_2\},\{q_0,q_2\},\{q_1,q_2\},\{q_0,q_1,q_2\}\}$,
  δ' next slide

NFA – DFA

Example (cont.):

- δ:

$$\begin{array}{cc} \delta(q_0, a) = \{q_0, q_1\}, & \delta(q_0, b) = \{q_1\}, \\ \delta(q_1, a) = \emptyset, & \delta(q_1, b) = \{q_2\}, \\ \delta(q_2, a) = \{q_0, q_1, q_2\}, & \delta(q_2, b) = \{q_1\}. \end{array}$$

- δ':

$$\begin{array}{cc} \delta'((\emptyset, a) = \emptyset, & \delta'((\emptyset, b) = \emptyset, \\ \delta'((\{q_0\}, a) = \{q_0, q_1\}, & \delta'((\{q_0\}, b) = \{q_1\}, \\ \delta'((\{q_1\}, a) = \emptyset, & \delta'((\{q_1\}, b) = \{q_2\}, \\ \delta'((\{q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_2\}, b) = \{q_1\}, \\ \delta'((\{q_0, q_1\}, a) = \{q_0, q_1\}, & \delta'((\{q_0, q_1\}, b) = \{q_1, q_2\}, \\ \delta'((\{q_0, q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_0, q_2\}, b) = \{q_1\}, \\ \delta'((\{q_1, q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_1, q_2\}, b) = \{q_1, q_2\}, \\ \delta'((\{q_0, q_1, q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_0, q_1, q_2\}, b) = \{q_1, q_2\}. \end{array}$$

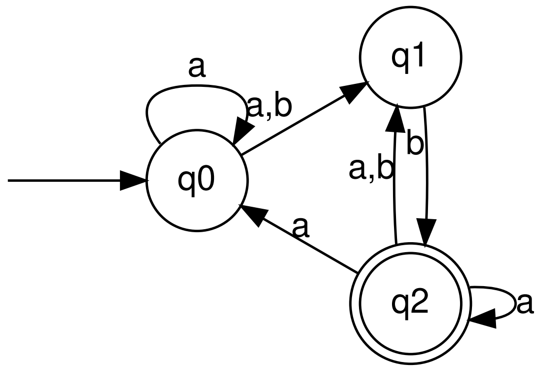
NFA – DFA

Example (cont.):

NVA

$$\begin{array}{cc} \delta(q_0,a)=\{q_0,q_1\}, & \delta(q_0,b)=\{q_1\}, \\ \delta(q_1,a)=\emptyset, & \delta(q_1,b)=\{q_2\}, \\ \delta(q_2,a)=\{q_0,q_1,q_2\}, & \delta(q_2,b)=\{q_1\}. \end{array}$$

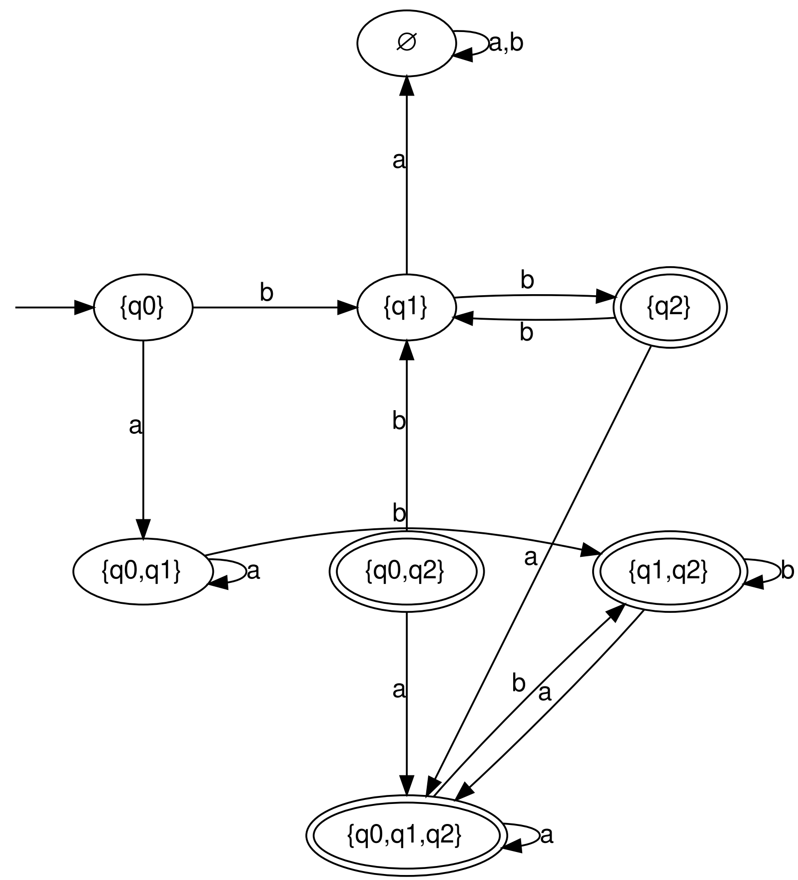
$F = \{q_2\}$



DVA

$$\begin{array}{cc} \delta'((\emptyset,a)=\emptyset, & \delta'((\emptyset,b)=\emptyset, \\ \delta'((\{q_0\},a)=\{q_0,q_1\}, & \delta'((\{q_0\},b)=\{q_1\}, \\ \delta'((\{q_1\},a)=\emptyset, & \delta'((\{q_1\},b)=\{q_2\}, \\ \delta'((\{q_2\},a)=\{q_0,q_1,q_2\}, & \delta'((\{q_2\},b)=\{q_1\}, \\ \delta'((\{q_0,q_1\},a)=\{q_0,q_1\}, & \delta'((\{q_0,q_1\},b)=\{q_1,q_2\}, \\ \delta'((\{q_0,q_2\},a)=\{q_0,q_1,q_2\}, & \delta'((\{q_0,q_2\},b)=\{q_1\}, \\ \delta'((\{q_1,q_2\},a)=\{q_0,q_1,q_2\}, & \delta'((\{q_1,q_2\},b)=\{q_1,q_2\}, \\ \delta'((\{q_0,q_1,q_2\},a)=\{q_0,q_1,q_2\}, & \delta'((\{q_0,q_1,q_2\},b)=\{q_1,q_2\}. \end{array}$$

$F' = \{\{q_2\},\{q_0,q_2\},\{q_1,q_2\},\{q_0,q_1,q_2\}\}$



Computing power of NFA

Lemma 1:

- For all $p,q \in Q$, $q' \in Q'$ és $u,v \in T^*$,
  if $qu \Rightarrow^*_A pv$ and $q \in q'$,
  then $\exists\, p' \in Q'$, s.t.
  $q'u \Rightarrow^*_{A'} p'v$ and $p \in p'$.

Proof:

- Induction over the number of reduction steps n in $qu \Rightarrow^*_A pv$.
- For $n=0$: the claim holds trivially, $p'=q'$.

Computing power of NFA

Proof (Lemma 1, cont.):

- For $n \to n+1$: Assume, the claim holds for all reductions of $\leq n$ steps.
- Let $qu \Rightarrow^*_A pv$ be a reduction of $n + 1$ steps.
  Then for some $q_1 \in Q$ and $u_1 \in T^*$ holds that
  $qu \Rightarrow_A q_1u_1 \Rightarrow^*_A pv$.
- Therefore, $\exists\ a \in T$, s.t. $u = au_1$ and $q_1 \in \delta(q, a)$.
- Since $\delta(q, a) \subseteq \delta'(q', a)$, for $q \in q'$,
  q'_1 can be choosen as $q'_1 = \delta'(q', a)$.
- Consequently, $q'u \Rightarrow_{A'} q'_1u_1$, where $q_1 \in q'_1$.
- By the induction assumption,
  $\exists\ p' \in Q'$, s.t. $q'_1u_1 \Rightarrow^*_{A'} p'v$ and $p \in p'$, which proves the claim. $\square$

Computing power of NFA

Proof (Theorem, cont.):

- Let $u \in L(A)$, i.e. $q_0 u \Rightarrow^*_A p$, for some $q_0 \in Q_0$, $p \in F$.
- By Lemma 1, $\exists\, p'$, s.t. $q'_0 u \Rightarrow^*_{A'} p'$, where $p \in p'$.
- By definition of F', $p \in p'$ and $p \in F$ imply that $p' \in F'$, which proves $L(A) \subseteq L(A')$.
- For $L(A') \subseteq L(A)$, we prove Lemma 2.

Computing power of NFA

Lemma 2:

- For all $p', q' \in Q'$, $p \in Q$ and $u, v \in T^*$,
  - if $q'u \Rightarrow^*_{A'} p'v$ and $p \in p'$,
  - then $\exists\, q \in Q$, s.t. $qu \Rightarrow^*_A pv$ and $q \in q'$.

Proof:

- Induction over the number of steps n in the reduction.
- For $n = 0$: The claim holds trivially.

Computing power of NFA

Proof (Lemma 2, cont.):

- For $n \rightarrow n+1$: Assume, the claim holds for all reductions of $\leq n$ steps.
- Let $q'u \Rightarrow^*_{A'} p'v$ be a reduction of $n + 1$ steps. Then $q'u \Rightarrow^*_{A'} p'_1 v_1 \Rightarrow_{A'} p'v$, where $v_1 = av$, for some $p'_1 \in Q'$ and $a \in T$.
- Then, $p \in p' = \delta'(p'_1, a) = \bigcup_{p_1 \in p'_1} \delta(p_1, a)$.
- Consequently, $\exists\, p_1 \in p'_1$, s.t. $p \in \delta(p_1, a)$.
- For this p_1, it holds that $p_1 v_1 \Rightarrow_A pv$.
- By the induction assumption, $qu \Rightarrow^*_A p_1 v_1$, for some $q \in q_0$, which implies the claim. $\square$

Computing power of NFA

Proof (Theorem, cont.):

- Let $q'_0 u \Rightarrow^*_{A'} p'$ and $p' \in F$.
- By the definition of F', $\exists\, p \in p'$, s.t. $p \in F$.
- Then, by Lemma 2, for some $q_0 \in q'_0$, holds that $q_0 u \Rightarrow^*_A p$.
- This proves the claim of the theorem. $\square$

Corollaries

Corollary 1:

- The class of regular languages $\mathcal{L}_3$ is closed for the complement operation.

Proof:

- Let L be a language, recognized by a FA $A = (Q,T,\delta,q_0,F)$
- Then $\overline{L} = T^* - L$ can be recognized by a FA $A = (Q,T,\delta,q_0,Q-F)$