Models of Computation

5: Regular expressions, finite automaton

Regular expressions

Applications

- search and replace dialogs of text editors
- search engines
- text processing utilities (e.g. sed and AWK)
- programming languages, lexical analysis
- genom analysis (genom as string)
- spam/malware filter
- ...

Regular expressions

Let V and $V' = \{\emptyset, \varepsilon, \cdot, +, *, (,)\}$ be disjoint alphabets. A **regular expression** over V is defined recursively as follows:

1. $\emptyset$ is a regular expression over V,
2. ε is a regular expression over V,
3. all $a \in V$ are regular expressions over V,
4. If R is a regular expression over V, then R^* is also a regular expression over V,
5. If Q and R are regular expressions over V, then $(Q \cdot R)$ and $(Q + R)$ are also regular expressions over V.

* denotes the closure of iteration,
· concatenation, and
\+ union.

Regular expressions

Each regular expression **represents a regular language**, which is defined as:

1. $\emptyset$ represents the empty language,
2. ε represents the language {ε},
3. Letter a ($\in V$) represents the language $\{a\}$,
4. if R is a regular expression over V, which represents the language L, then R^* represents L^*,
5. if R and R' are regular expressions over V, s.t.
   R represents the language L and
   R' represents the language L', then
   $(R \cdot R')$ represents the language LL',
   $(R + R')$ represents the language $L \cup L'$.

Regular expressions

Parentheses can be omitted when defining precedence on operations. The the usual sequence is: *, ·, +. The following regular expressions are equivalent:

- a^* is the same as $(a)^*$ and represent the language $\{a\}^*$.
- $(a + b)^*$ is the same as $((a) + (b))^*$ and represents the language $\{a, b\}^*$.
- $a^* \cdot \mathrm{b}$ is the same as $((a)^*) \cdot (b)$ and represents the language $\{a\}^*b$.
- $b + ab^*$ is the same as $(b) + ((a) \cdot (b)^*)$ and represents the language $\{b\} \cup \{a\}\{b\}^*$.
- $(a + b) \cdot a^*$ is the same as $((a) + (b)) \cdot ((a)^*)$ and represents the language $\{a, b\}\{a\}^*$.

Regular expressions

Let P, Q, an R be regular expressions. Then following hold:

- $P + (Q + R) = (P + Q) + R$
- $P \cdot (Q \cdot R) = (P \cdot Q) \cdot R$
- $P + Q = Q + P$
- $P \cdot (Q + R) = P \cdot Q + P \cdot R$
- $(P + Q) \cdot R = P \cdot R + Q \cdot R$
- $P^* = \varepsilon + P \cdot P^*$
- $\varepsilon \cdot P = P \cdot \varepsilon = P$
- $P^* = (\varepsilon + P)^*$

Regular expressions

Example:

The language represented the regular expressions
$(a + b)a^*$ and $aa^* + ba^*$ is the same:
$\{ aa^n \mid n \geq 0 \} \cup \{ ba^n \mid n \geq 0 \}$.

The language represented by $a + ba^*$ is:
$\{ a, b, ba, ba^2, ba^3, \ldots \}$.

Expressive power of regular expressions

Theorem:

1) Every regular expression represents a regular (type 3) language.

2) For every regular (type 3) language, there is a regular expression representing the language.

Proof:

1) follows from the fact that the class of regular languages $\mathcal{L}_3$ is closed for the regular operations.

Expressive power of regular expressions

Proof:

For 2), we show that for every regular language L generated by a grammar $G = (N, T, P, S)$, a regular expression can be constructed, that represents L.

- Let $N = \{A_1, \ldots, A_n\}$, $n \geq 1$, $S = A_1$.
- Assume, each rule of G is of form $A_i \to aA_j$ or $A_i \to \varepsilon$, where $a \in T$, $1 \leq i, j \leq n$.
- We say that a non-terminal A_m is **affected** by the derivation $A_i \Rightarrow^* uA_j$ ($u \in T^*$), if A_m occurs in an intermediate string between A_i and uA_j in the derivation.

Expressive power of regular expressions

Proof (cont.):

- A derivation $A_i \Rightarrow^* uA_j$ is called ***k*-bounded** if $0 \leq m \leq k$ holds for all non-terminals A_m occurring in the derivation.
- Let $E^k_{i,j} = \{u \in T^* \mid \exists\, A_i \Rightarrow^* uA_j$ k-bounded derivation$\}$.
- We show by induction on k, that for language $E^k_{i,j}$, there is a regular expression representing $E^k_{i,j}$, $0 \leq i,j,k \leq n$.

Expressive power of regular expressions

Proof (cont.):

- $k = 0$ (induction start):
  - For $i \neq j$, $E^{0}_{i,j}$ is eighter empty, or it consists of symbols of T ($a \in E^{0}_{i,j}$ if and only if $A_i \rightarrow aA_j \in P$.)
  - For $i = j$, $E^{0}_{i,j}$ consists of ε and zero or more elements of T, so $E^{0}_{i,j}$ can be represented by a regular expression.

Expressive power of regular expressions

Proof (cont.):

- k-1 → k (induction step):
  - Assume that for a fixed k, $0 < k \leq n$, $E^{k-1}_{i,j}$ can be represented by a regular expression.
  - Then for all i, j, k it holds that $E^k_{i,j} = E^{k-1}_{i,j} + E^{k-1}_{i,k} \cdot (E^{k-1}_{k,k})^* \cdot E^{k-1}_{k,j}$.
  - Therefore, $E^k_{i,j}$ can also be represented by a regular expression.
  - Let I_ε be the set of indices i for which $A_i \rightarrow \varepsilon$.
  - Then $L(G) = \bigcup_{i \in I_\varepsilon} E^n_{1,i}$ can be representd by a regular expression. The claim of the theorem follows. □

Finite Automata (FA)

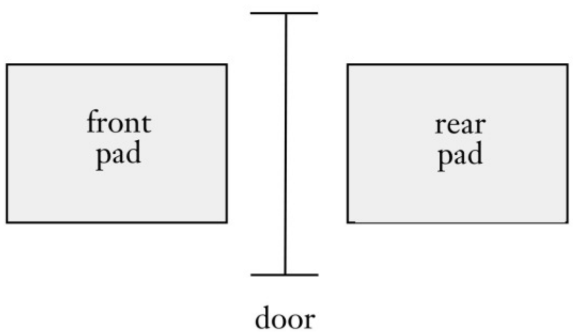
- Identifying formal languages is also possible with recognition devices, i.e. by automata.
- An automaton can process and identify words.
- Grammars use a synthesizing approach, while automata an analytic one.
- An automaton accepts or rejects an input word.

Finite Automata (FA)

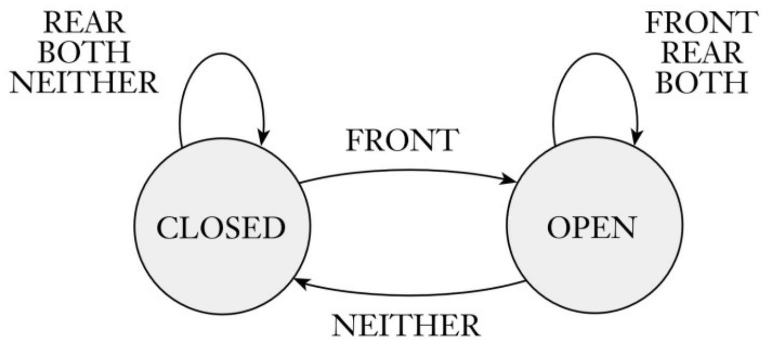
- A finite automaton (FA) performs a sequence of steps in discrete time intervals
- The FA starts in the initial state.
- The input word is located on the input tape and the reading head is on the leftmost symbol of an input word.
- After reading a symbol, the FA moves the reading head to one position to the right, then the state changes, regarding the state transition function.
- If the FA has read the input, it stops, accepts or rejects the input.

Finite Automata (FA)

- Example: automatic door control



State transition diagram:



State transition table:

input signal

state		NEITHER	FRONT	REAR	BOTH
	CLOSED	CLOSED	OPEN	CLOSED	CLOSED
	OPEN	CLOSED	OPEN	OPEN	OPEN

Finite Automata (FA)

- Application examples:
  - Automatic door control
  - Coffee machine
  - Pattern recognition
  - Markov chains
  - Pattern recognition
  - Speech processing
  - Optical character recognition
  - Predictions of share prizes in the stock exchange
  - ...

Finite Automata (FA)

A finite automaton is a 5-tuple $A = (Q, T, \delta, q_0, F)$, where

- Q is a finite, nonempty set of **states**,
- T is the finite **alphabet of input symbols**,
- $\delta : Q \times T \rightarrow Q$ is the **state transition function**
- $q_0 \in Q$ is the **initial state** or **start state**,
- $F \subseteq Q$ is the set of **acceptance states** or **end states**.

Finite Automata (FA)

Remark:

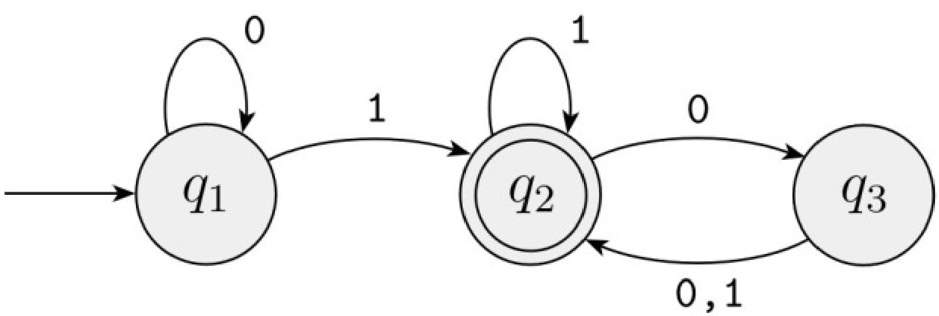
- The function δ can be extended to a function $\hat{\delta} : Q \times T^* \to Q$ as follows:
  - $\hat{\delta}(q, \varepsilon) = q$,
  - $\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$ for all $x \in T^*$ and $a \in T$.

Finite Automata (FA)

Example:

- Let $A = (Q, T, \delta, q_1, F)$ be a FA, where
  $Q = \{q_1, q_2, q_3\}$, $T = \{0, 1\}$, $F = \{q_2\}$, and
  $\delta(q_1, 0) = q_1$, $\delta(q_1, 1) = q_2$, $\delta(q_2, 0) = q_3$, $\delta(q_2, 1) = q_2$,
  $\delta(q_3, 0) = \delta(q_3, 1) = q_2$.

State transition diagram:



State transition table:

δ	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

- The accepted language is $L(A)=\{w \mid w$ conains at least one 1 and the last 1 is not followed by an odd number of 0s$\}$

Finite Automata (FA)

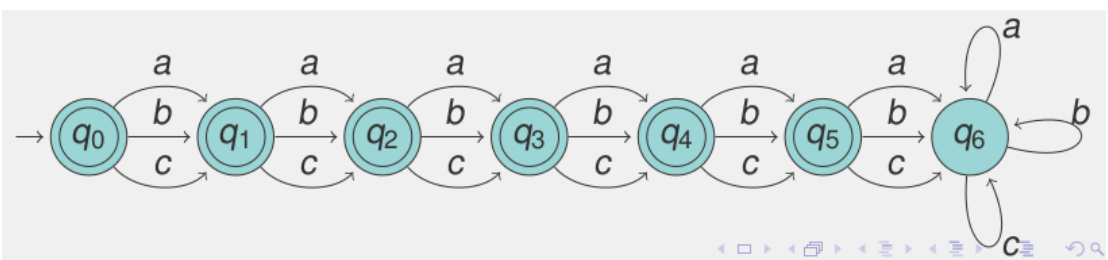
Example:

- Let $T = \{a,b,c\}$.
  Define a FA, which accepts the words of length of at most 5.

Solution:

- Formaly:
  $A=(\{q_0, \ldots, q_6\}, \{a, b, c\}, \delta, q_0, \{q_0, \ldots, q_5\})$,
  $\delta(q_i, t) = q_{i+1}$, for $i = 0, \ldots, 5$, $t \in \{a, b, c\}$,
  $\delta(q_6, t) = q_6$, for $t \in \{a, b, c\}$

- State transition diagram:

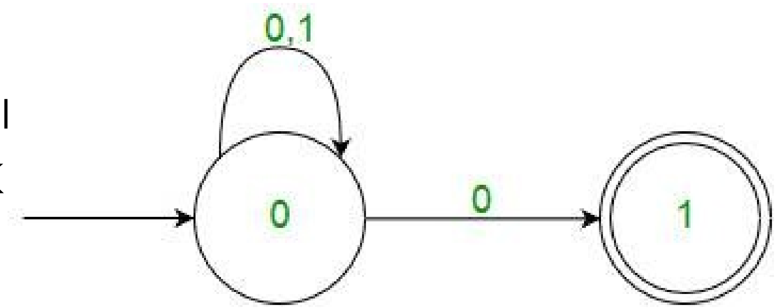


State transition table:

	a	b	c
$\leftrightarrows q_0$	q_1	q_1	q_1
$\leftarrow q_1$	q_2	q_2	q_2
$\leftarrow q_2$	q_3	q_3	q_3
$\leftarrow q_3$	q_4	q_4	q_4
$\leftarrow q_4$	q_5	q_5	q_5
$\leftarrow q_5$	q_6	q_6	q_6
q_6	q_6	q_6	q_6

Deterministic and non-deterministic finite automata

- **Deterministic finite automaton (DFA)**:
  Function δ is single-valued, i.e. $\forall\ (q, a) \in Q \times T$ there is exactly one state s, s.t. $\delta(q, a) = s$.
- **Nondeterministic finite automaton (NFA)**:
  - Function δ is multi-valued, i.e. $\delta : Q \times T \rightarrow 2^Q$.
  - Multiple initial states are allowed (the set of initial states $Q_0 \subseteq Q$).
  - It is allowed that $\delta(q, a) = \emptyset$ for so (q,a), i.e. the machine gets stuck
  - Null (or ε) move is allowed, i.e. it can move forward without reading symbols.



NFA example

Deterministic and non-deterministic FA

- New features of non-determinism
  - Multiple paths are possible (multiple choises at each step).
  - ε-transition is a “free” move without reading input.
  - Accepts the input if <u>some</u> path leads to an accepting state.

Deterministic and non-deterministic FA

- Alternative notation:
- State transitions can also be given in the form $qa \rightarrow p$, where $p \in \delta(q, a)$.
- Let M_δ be set of rules of the state transition of an NFA $A = (Q, T, \delta, Q_0, F)$.
- If M_δ contains exactly one rule $qa \rightarrow p$ for each pair (q,a), then the FA is deterministic, oherwise non-deterministic.

FA – reduction

- Let $A = (Q, T, \delta, Q_0, F)$ be a FA and $u,v \in QT^*$ words. The FA A **reduces** the u **in one step** (**directly**) to v (notation: $u \Rightarrow_A v$, or short: $u \Rightarrow v$), if there are a rule $qa \to p \in M_\delta$ (i.e. $\delta(q, a) = p$) and a word $w \in T^*$, s.t. $u = qaw$ and $v = pw$ hold.
- The FA $A = (Q, T, \delta, Q_0, F)$ **reduces** $u \in QT^*$ to $v \in QT^*$ (notation: $u \Rightarrow_A^* v$, or short: $u \Rightarrow^* v$, if
  - either $u = v$,
  - or $\exists$ a word $z \in QT^*$, s.t. $u \Rightarrow^* z$ and $z \Rightarrow v$.

- Remark: $\Rightarrow^*$ is the reflexive, transitive closure of $\Rightarrow$.

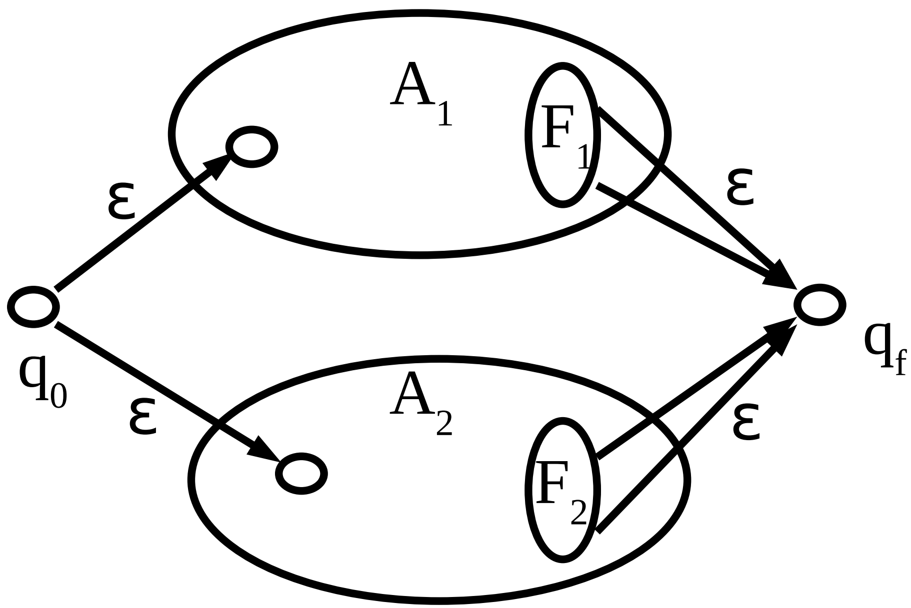
FA – accepted language

- The **language accepted/recognized** by the FA $A = (Q, T, \delta, Q_0, F)$ is:
  $L(A) = \{u \in T^* \mid q_0u \Rightarrow^* p$ for some $q_0 \in Q_0$ and $p \in F\}$
- For a DFA A, there is one single start state $Q_0=\{q_0\}$. The language accepted by DFA A is:
  $L(A) = \{u \in T^* \mid q_0u \Rightarrow^* p$ for some $p \in F\}$

NFA accepting L_1 U L_2

Theorem: If L_1 and L_2 are regular languages, then L_1U L_2 is also a regular language.

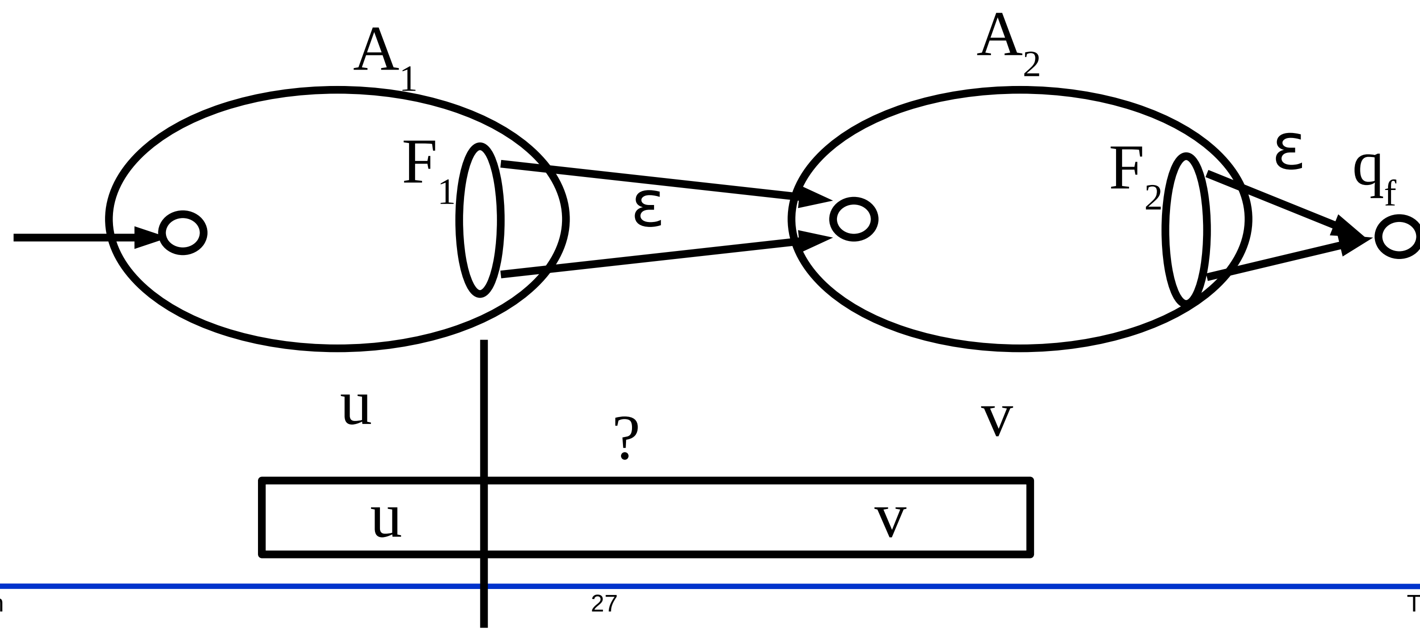
Proof (sketch): Let A_1 be a DFA, accepting L_1 and A_2 a DFA accepting L_2. Then the following NFA accepts L_1U L_2.



NFA accepting $L_1 L_2$

Theorem: If L_1 and L_2 regular languages, then $L_1 L_2$ is also a regular language.

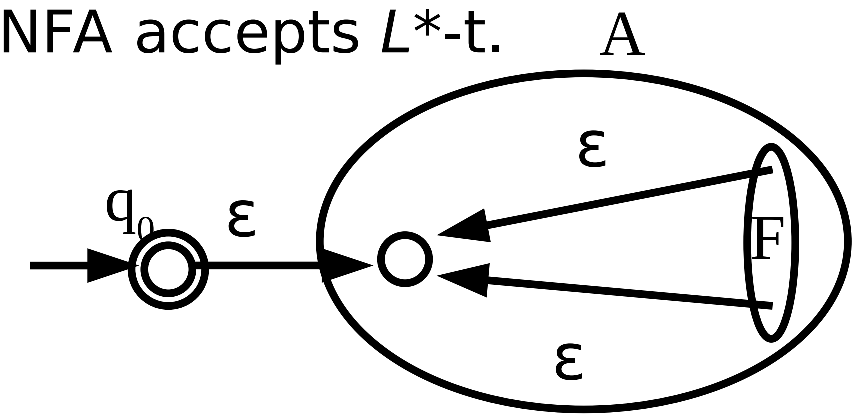
Proof (sketch): Let A_1 be a DFA accepting L_1,
A_2 egy DFA accepting L_2.
The following NVA accepts $L_1 L_2$.

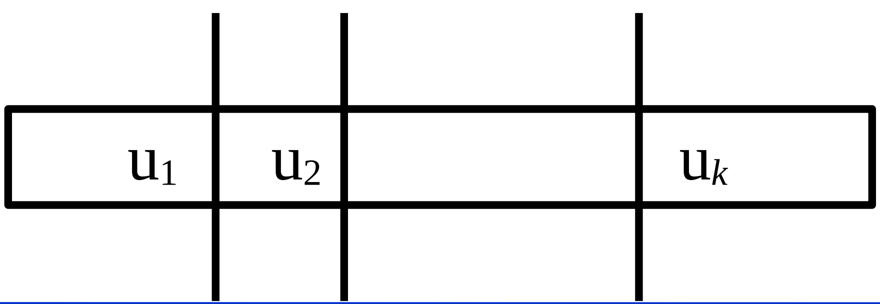


NFA accepting *L**

Theorem: If *L* is a regular language, then *L** is also a regular language.

Proof. (sketch): Let *A* be a DFA accepting *L.*
The fillowing NFA accepts *L**-t.





Computing power of NFA

- **Theorem**: For all NFA $A = (Q, T, \delta, Q_0, F)$ a DFA $A' = (Q', T, \delta', q'_0, F')$ can be constructed, s.t. $L(A) = L(A')$ holds.

- Idea: DFA keeps track of the subset of possible states in NFA
- Remark: In worst case $|Q'| = 2^{|Q|}$.

Computing power of NFA

Proof:

- Let $Q' = 2^Q$ be the set of all subsets of the set Q.
(the number of elements of Q' is $2^{|Q|}$).
- Let $\delta' : Q' \times T \rightarrow Q'$ be the function defined as:
$\delta'(q', a) = \bigcup_{q \in q'} \delta(q, a)$.
- Let $q'_0 = Q_0$ and $F' = \{q' \in Q' \mid q' \cap F \neq \emptyset\}$
- To prove $L(A) \subseteq L(A')$ we prove Lemma 1,
to $L(A') \subseteq L(A)$ we prove Lemma 2.
- First, an example (next slide)

NFA – DFA

Example:

- Let $A = (Q,T,\delta,Q_0,F)$ be a NFA, where
  $Q = \{q_0, q_1, q_2\}$, $T = \{a, b\}$, $Q_0 = \{q_0\}$, $F = \{q_2\}$.
  δ is defined as:
  $\delta(q_0, a) = \{q_0, q_1\}$, $\delta(q_0, b) = \{q_1\}$,
  $\delta(q_1, a) = \emptyset$, $\delta(q_1, b) = \{q_2\}$,
  $\delta(q_2, a) = \{q_0, q_1, q_2\}$, $\delta(q_2, b) = \{q_1\}$.
  Construct a DFA A' quivalent with A.

Solution:

- DFA: $A' = (Q',T,\delta',q'_0,F')$, where
  $Q' = \{\emptyset,\{q_0\},\{q_1\},\{q_2\},\{q_0, q_1\},\{q_0,q_2\},\{q_1,q_2\},\{q_0,q_1,q_2\}\}$,
  $q'_0 = \{q_0\}$,
  $F' = \{\{q_2\},\{q_0,q_2\},\{q_1,q_2\},\{q_0,q_1,q_2\}\}$,
  δ' next slide

NFA – DFA

Example (cont.):

- δ:

$$\begin{array}{cc} \delta(q_0, a) = \{q_0, q_1\}, & \delta(q_0, b) = \{q_1\}, \\ \delta(q_1, a) = \emptyset, & \delta(q_1, b) = \{q_2\}, \\ \delta(q_2, a) = \{q_0, q_1, q_2\}, & \delta(q_2, b) = \{q_1\}. \end{array}$$

- δ':

$$\begin{array}{cc} \delta'((\emptyset, a) = \emptyset, & \delta'((\emptyset, b) = \emptyset, \\ \delta'((\{q_0\}, a) = \{q_0, q_1\}, & \delta'((\{q_0\}, b) = \{q_1\}, \\ \delta'((\{q_1\}, a) = \emptyset, & \delta'((\{q_1\}, b) = \{q_2\}, \\ \delta'((\{q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_2\}, b) = \{q_1\}, \\ \delta'((\{q_0, q_1\}, a) = \{q_0, q_1\}, & \delta'((\{q_0, q_1\}, b) = \{q_1, q_2\}, \\ \delta'((\{q_0, q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_0, q_2\}, b) = \{q_1\}, \\ \delta'((\{q_1, q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_1, q_2\}, b) = \{q_1, q_2\}, \\ \delta'((\{q_0, q_1, q_2\}, a) = \{q_0, q_1, q_2\}, & \delta'((\{q_0, q_1, q_2\}, b) = \{q_1, q_2\}. \end{array}$$

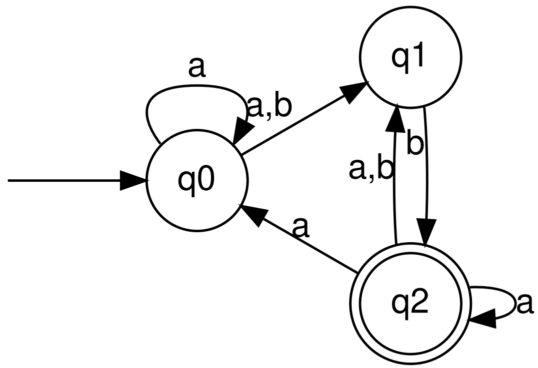
NFA – DFA

Example (cont.):

NFA

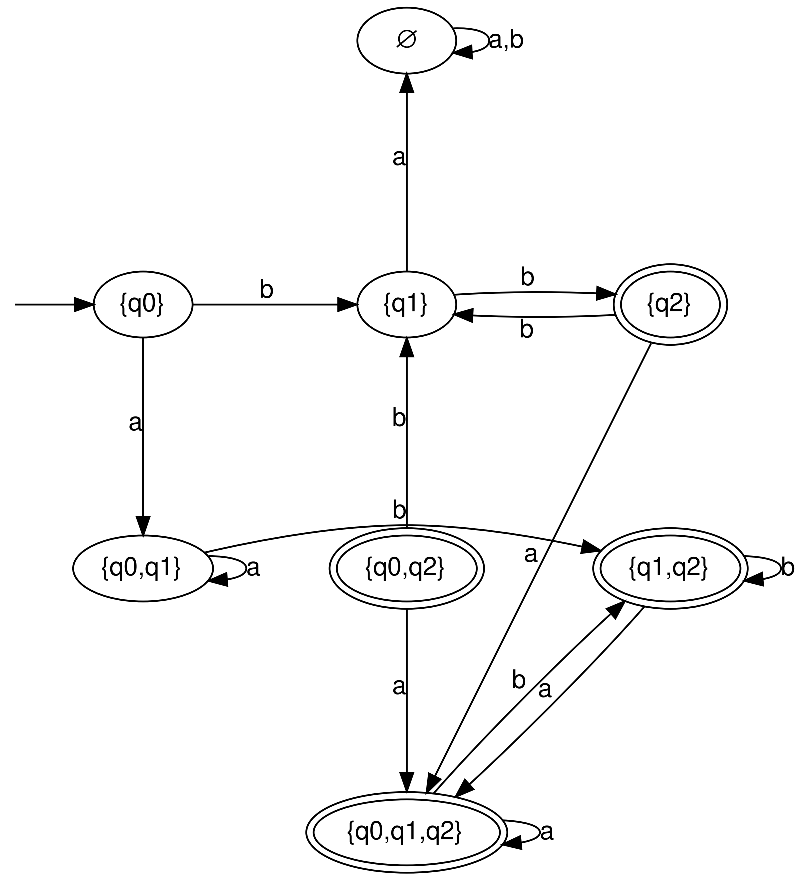
$$\delta(q_0,a)=\{q_0,q_1\},\quad \delta(q_0,b)=\{q_1\},$$
$$\delta(q_1,a)=\emptyset,\quad \delta(q_1,b)=\{q_2\},$$
$$\delta(q_2,a)=\{q_0,q_1,q_2\},\quad \delta(q_2,b)=\{q_1\}.$$

$F = \{q_2\}$



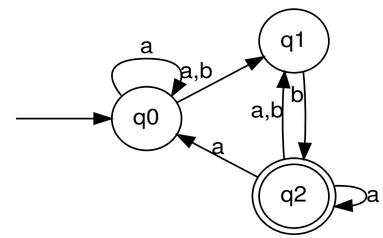
DFA

$$\delta'((\emptyset,a)=\emptyset,\quad \delta'((\emptyset,b)=\emptyset,$$
$$\delta'((\{q_0\},a)=\{q_0,q_1\},\quad \delta'((\{q_0\},b)=\{q_1\},$$
$$\delta'((\{q_1\},a)=\emptyset,\quad \delta'((\{q_1\},b)=\{q_2\},$$
$$\delta'((\{q_2\},a)=\{q_0,q_1,q_2\},\quad \delta'((\{q_2\},b)=\{q_1\},$$
$$\delta'((\{q_0,q_1\},a)=\{q_0,q_1\},\quad \delta'((\{q_0,q_1\},b)=\{q_1,q_2\},$$
$$\delta'((\{q_0,q_2\},a)=\{q_0,q_1,q_2\},\quad \delta'((\{q_0,q_2\},b)=\{q_1\},$$
$$\delta'((\{q_1,q_2\},a)=\{q_0,q_1,q_2\},\quad \delta'((\{q_1,q_2\},b)=\{q_1,q_2\},$$
$$\delta'((\{q_0,q_1,q_2\},a)=\{q_0,q_1,q_2\},\quad \delta'((\{q_0,q_1,q_2\},b)=\{q_1,q_2\}.$$



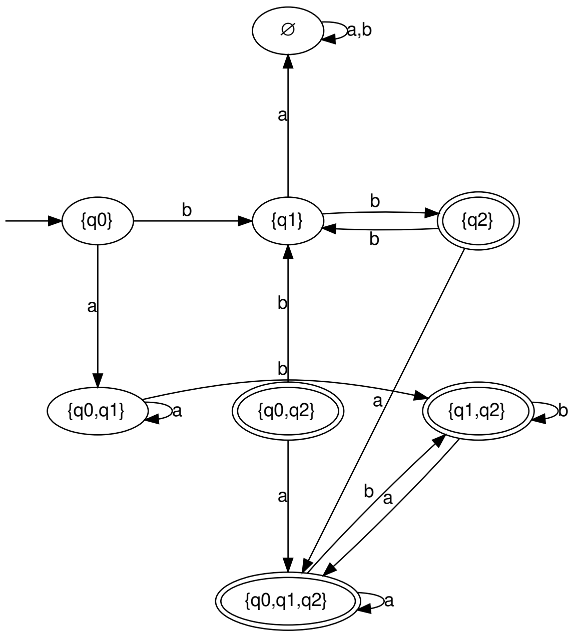
$F' = \{\{q_2\},\{q_0,q_2\},\{q_1,q_2\},\{q_0,q_1,q_2\}\}$

Computing power of NFA



Lemma 1:

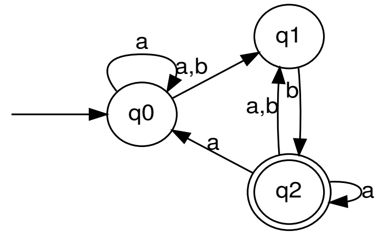
- For all $p,q \in Q$, $q' \in Q'$ and $u,v \in T^*$, if $qu \Rightarrow^*_A pv$ and $q \in q'$, then $\exists\, p' \in Q'$, s.t. $q'u \Rightarrow^*_{A'} p'v$ and $p \in p'$.



Proof:

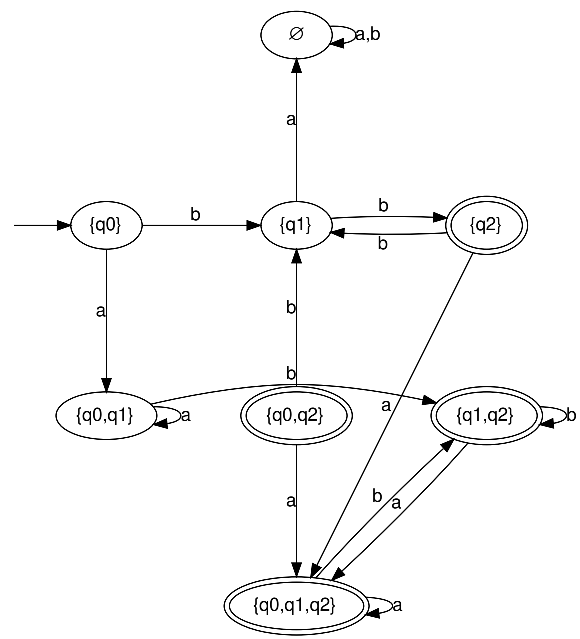
- Induction over the number of reduction steps n in $qu \Rightarrow^*_A pv$.
- For n=0: the claim holds trivially, $p'=q'$

Computing power of NFA



Proof (Lemma 1, cont.):

- For $n \to n{+}1$: Assume, the claim holds for all reductions of $\leq n$ steps.
- Let $qu \Rightarrow^*_A pv$ be a reduction of $n + 1$ steps. Then for some $q_1 \in Q$ and $u_1 \in T^*$ holds that $qu \Rightarrow_A q_1u_1 \Rightarrow^*_A pv$.
- Therefore, $\exists\ a \in T$, s.t. $u = au_1$ and $q_1 \in \delta(q, a)$.
- Since $\delta(q, a) \subseteq \delta'(q', a)$, for $q \in q'$, q'_1 can be choosen as $q'_1 = \delta'(q', a)$.
- Consequently, $q'u \Rightarrow_{A'} q'_1u_1$, where $q_1 \in q'_1$.
- By the induction assumption, $\exists\ p' \in Q'$, s.t. $q'_1u_1 \Rightarrow^*_{A'} p'v$ and $p \in p'$, which proves the claim. $\square$

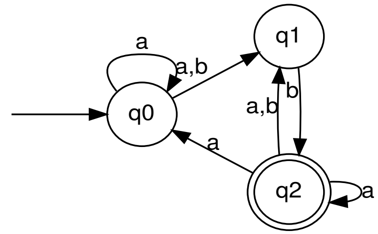


Computing power of NFA

Proof (Theorem, cont.):

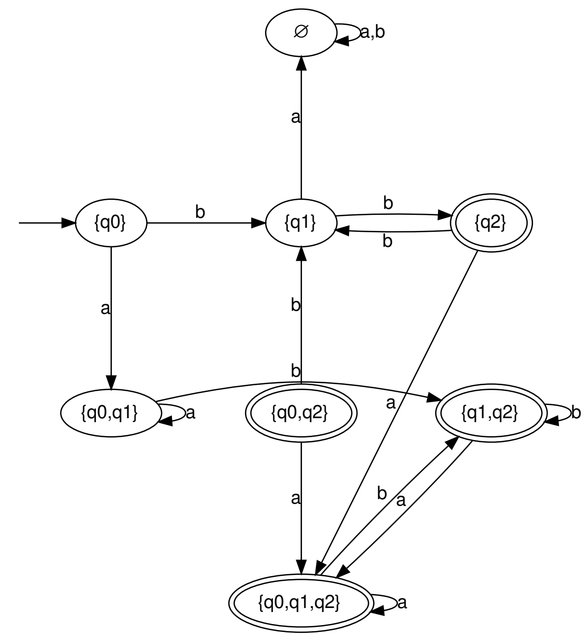
- Let $u \in L(A)$, i.e. $q_0 u \Rightarrow^*_A p$, for some $q_0 \in Q_0$, $p \in F$.
- By Lemma 1, $\exists\, p'$, s.t. $q'_0 u \Rightarrow^*_{A'} p'$, where $p \in p'$.
- By definition of F', $p \in p'$ and $p \in F$ imply that $p' \in F'$, which proves $L(A) \subseteq L(A')$.
- For $L(A') \subseteq L(A)$, we prove Lemma 2.

Computing power of NFA



Lemma 2:

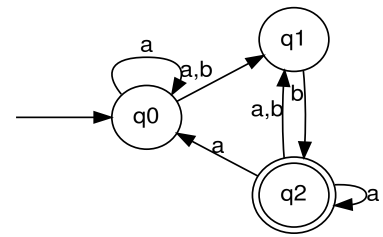
- For all $p', q' \in Q', p \in Q$ and $u, v \in T^*$,
    - if $q'u \Rightarrow^*_{A'} p'v$ and $p \in p'$,
    - then $\exists\, q \in Q$, s.t.
      $qu \Rightarrow^*_{A} pv$ and $q \in q'$.



Proof:

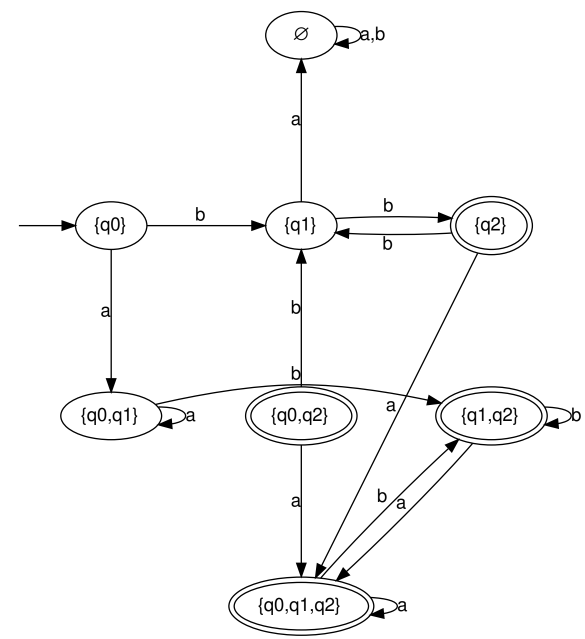
- Induction over the number of reduction steps n.
- For $n = 0$: The claim holds trivially.

Computing power of NFA



Proof (Lemma 2, cont.):

- For $n \rightarrow n + 1$: Assume, the claim holds for all reductions of $\leq n$ steps.
- Let $q'u \Rightarrow^*_{A'} p'v$ be a reduction of $n + 1$ steps. Then $q'u \Rightarrow^*_{A'} p'_1 v_1 \Rightarrow_{A'} p'v$, where $v_1 = av$, for some $p'_1 \in Q'$ and $a \in T$.
- Then, $p \in p' = \delta'(p'_1, a) = \bigcup_{p_1 \in p'_1} \delta(p_1, a)$.
- Consequently, $\exists\, p_1 \in p'_1$, s.t. $p \in \delta(p_1, a)$.
- For this p_1, it holds that $p_1 v_1 \Rightarrow_A pv$.
- By the induction assumption, $qu \Rightarrow^*_A p_1 v_1$, for some $q \in q_0$, which implies the claim. $\square$



Computing power of NFA

Proof (Theorem, cont.):

- Let $q'_0 u \Rightarrow^*_{A'} p'$ and $p' \in F$.
- By the definition of F', $\exists\ p \in p'$, s.t. $p \in F$.
- Then, by Lemma 2, for some $q_0 \in q'_0$, holds that $q_0 u \Rightarrow^*_A p$.
- This proves the claim of the theorem. $\square$

Corollaries

Corollary 1:

- The class of regular languages $\mathcal{L}_3$ is closed for the complement operation.

Proof:

- Let L be a language, recognized by a FA $A = (Q,T,\delta,q_0,F)$
- Then $\overline{L} = T^* - L$ can be recognized by a FA $A = (Q,T,\delta,q_0,Q-F)$

Corollaries

Corollary 2:

- The class of regular languages $\mathcal{L}_3$ is closed for the intersection operation.

Proof:

- We know, that $\mathcal{L}_3$ is closed for the union operation.
- $L_1 \cap L_2 = \overline{\overline{L}_1 \cup \overline{L}_2}$.
- By Corollary 1, the claim follows.

FA – Myhill-Nerode Theorem

- Let L be a language over the alphabet T. The **relation E_L induced by language L** is a binary relation on T^*, for which it holds that

  $\forall\ u, v \in T^*$, uE_Lv, if and only if $\nexists\ w \in T^*$, s.t. exatly one of the words uw and vw is an element of L (i.e. $\forall\ w \in T^* : uw \in L$ if and only if $vw \in L$).

- E_L is an **equivalence relation** and it is **right-invariant**. (Right-invariant: if uE_Lv, then uwE_Lvw holds for every word $w \in T^*$.)
- The **index of the E_L** is the number of its equivalence classes.

Theorem (Myhill-Nerode): $L \subseteq T^*$ can be recognized by a deterministic FA if and only if E_L has a finite index.

FA – Myhill-Nerode Theorem

Theorem (Myhill-Nerode): $L \subseteq T^*$ can be recognized by a DFA if and only if E_L has a finite index.

- This index is equal to the number of states in the minimal DFA recognizing L.

DFA with minimum number of states

- The DFA A has a minimum number of states (**minimal DFA**), if there is no DFA A', which recognizes the same language as A, but the number of states of A' is smaller than the number of states of A.

Theorem: The minimal DFA accepting the regular language L is unique, up to isomorphism.

DFA with minimum number of states

Theorem: The minimal DFA accepting the regular language L is unique, up to isomorphism.

- Let $A = (Q, T, \delta, q_0, F)$ be a DFA. Define a relation $R \subseteq Q \times Q$, s.t. pRq if $\forall$ input word $x \in T^*$ it holds that $px \Rightarrow^*_A r$ if and only if $qx \Rightarrow^*_A r'$ for some $r, r' \in F$ states. ($r = r'$ is possible).
- States p and q are **distinguishable** if $\exists\, x \in T^*$, s.t. either $px \Rightarrow^*_A r$, $r \in F$, or $qx \Rightarrow^*_A r'$, $r' \in F$, but both reductions are not possible. Otherwise, p and q are **indistinguishable**.
- If p and q are indistinguishable, then $\delta(p, a) = s$ and $\delta(q, a) = t$ are indistinguishable for any $a \in T$.
- If $\delta(p, a) = s$ and $\delta(q, a) = t$ are distinguishable for $x \in T^*$, then p and q are distinguishable also for ax.

DFA with minimum number of states

- Let $A = (Q, T, \delta, q_0, F)$ be a DFA. State q is **reachable** from the initial state if there is a reduction $q_0x \Rightarrow^* q$, where x is some word over T.
- The DFA $A = (Q, T, \delta, q_0, F)$ is **connected**, if all its states are reachable from the initial state.
- We define the **set *H* of reachabele states** as follows:
  Let $H_0 = \{q_0\}$, $H_{i+1} = H_i \cup \{r \mid \delta(q, a) = r \text{ , } q \in H_i \text{ , } a \in T\}$, $i = 1, 2, \ldots$.
  Then $\exists\ k \geq 0 : H_k = H_l$, for all $l \geq k$. Let $H = H_k$.
- We define the DFA $A' = (Q', T, \delta', q_0, F')$ with
  $Q' = H$, $F' = F \cap H$ and $\delta' : H \times T \to H$ s.t. $\delta'(q, a) = \delta(q, a)$,
  if $q \in H$.
- It can be shown that A' is connected and accepts the same language as A. A' is the **largest connected subautomaton** of A.

DFA with minimum number of states

Computing_Reachable_States
(from: https://en.wikipedia.org/wiki/DFA_minimization)

- let reachable_states := {q0}
- let new_states := {q0}
- do {
-     temp := the empty set
-     for each q in new_states do
-         for each c in T do
-             temp := temp ∪ {p such that p = δ(q,c)}
-     new_states := temp \ reachable_states
-     reachable_states := reachable_states ∪ new_states
- } while (new_states ≠ the empty set)
- unreachable_states := Q \ reachable_states

DFA with minimum number of states

- Computing the minimal DFA
  (Hopcroft's partition refinement):
  - Determine, whether the automaton is connected or not.
    - If it is not connected, then consider the largest connected subautomaton.
      In the following, we assume, that the DFA is connected.
  - Partition the set of states according to distinguishability (states are divided into equivalence classes) (Steps 1-3)

DFA with minimum number of states

- **Step 1**:
  - Divide the set of states into two partitions: F and $Q - F$. (The states in F can be distinguished from the states in $Q - F$ by the empty word).
  - Repeat splitting of the partitions (Step 2) into additional partitions as long as the number of partitions remains the same.
- **Step 2**:
  - Consider an arbitrary partition P of states. Take an input symbol a and consider $\delta(p, a)$ for each state $p \in P$.
    If the obtained states belong to different partitions, then split P into as many new partitions as arosing in this way.
  - Perform this procedure for each input symbol and each partition, until no new partition is created.

DFA with minimum number of states

- **Step 3**:
  - Determine the DFA with the minimum number of states:
  - For each partition B_i, consider a representative state b_i.
  - Construct a DFA $A' = (Q', T, \delta', q_0, F')$, where
    - Q' is set of representatives of the partitions,
    - q'_0 is the representative of the partition containing q_0,
    - $\delta'(b_i, a) = b_j$, if $\exists\ q_i \in B_i$ and $q_j \in B_j$, s.t. $\delta(q_i, a) = q_j$.
    - $F' = \{b_f\}$ is the representative of the partition that contains the elements of F.

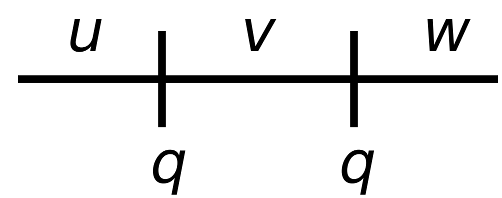
Pumping lemma for regular languages

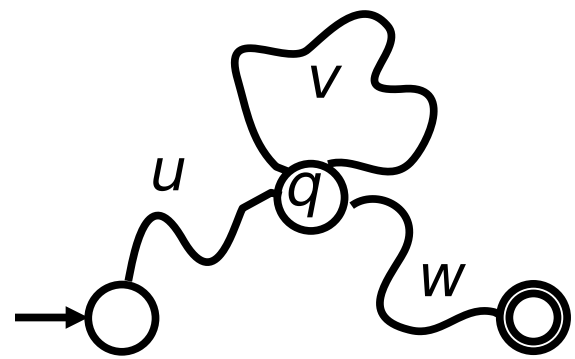
- A necessary condition for regular languages (i.e. recognizable by a FA).

- **Theorem** (pumping lemma for regular languages): For every regular language L there exists a natural number n, s.t. for all words $z \in L$ with $|z|>n$, holds that z can be written as $z=uvw$, satisfying the following conditions:
  1. $|uv| \leq n$,
  2. $|v|>0$,
  3. $uv^iw \in L$, for all $i \geq 0$.

Pumping lemma for regular languages

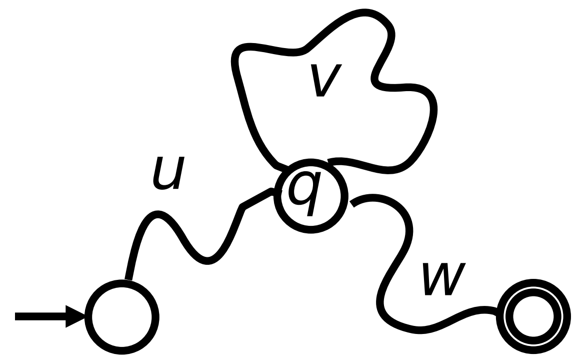
- **Proof.**:
  - Let L be a regular language and $A=(Q, T, \delta, q_0, F)$ be a minimal DFA, s.t. $L(A)=L$.
  - Let $n=|Q|+1$. Let $z \in L$ be an arbitrary word with $|z|>n$.
  - Consider A with input z. There must be a state q that A visits at least twice during the processing of z. Such a state q must already exist during the first n state transitions.
  - Let u be the prefix of z processed by A up to the first occurrence of q, and let v be the subword of z processed between the first and second occurrences of q. Then $|uv| \leq n$.

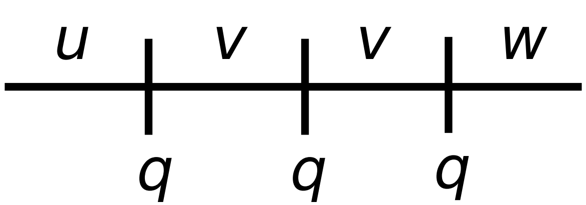




Pumping lemma for regular languages

- **Proof** (cont.):
  - Since at least one state transition has occurred in A between two occurrences of q, i.e. at least one symbol has been read, therefore $|v|>0$.
  - If A starts from the state q and reads the word w, it reaches the accepting end state. Accordingly, A accepts uw.
  - Similarly, A accepts all words of the form uv^iw, $i\geq 0$, since after reading u, A goes to state q, starting from q after reading v^i, A returns to state q, finally after reading w, A reaches an accepting end state. This completes the proof. $\square$

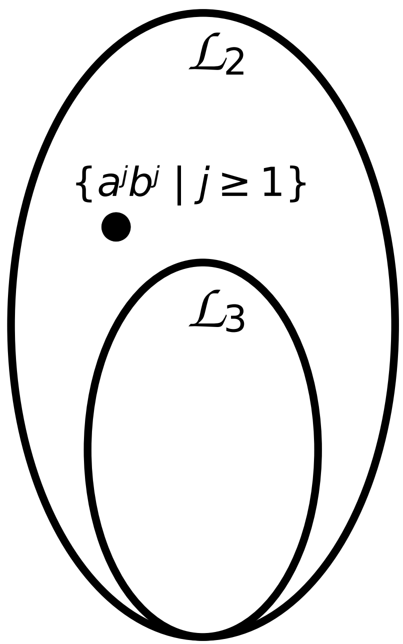




Application of the pumping lemma

Claim: The language $L=\{a^j b^j \mid j \geq 1\}$ is not regular.

Proof: Assume that G is a regular grammar generating L.
Then, by the regular pumping lemma,
$\exists\, n \geq 0$, s.t. $\forall\, z \in L$ words with $|z|>n$,
z can be written as $z=uvw$, satisfying
$|uv| \leq n$, $|v|>0$, and $uv^i w \in L$, for all $i \geq 0$.
Consider a word $a^m b^m$, where $m>n$.
Since $|uv| \leq n$, uv contains a symbols.
Since $|v|>0$, for $i \geq 2$, $uv^i w$ contains more
a symbols than b symbols.
Conseqently, $uv^i w \notin L$. $\square$



A context-free grammar generating L:
$S \to ab$, $S \to aSb$.

Transforming Regular Grammars to Equivalent FA

1) Construct an ε-free regular grammar G' from G (see next slide);

2) Create a FA M, with a state for every non-terminal in G'. Set the state representing the start symbol S' in G' to be the start state;

3) Add a new state F, which is final state;

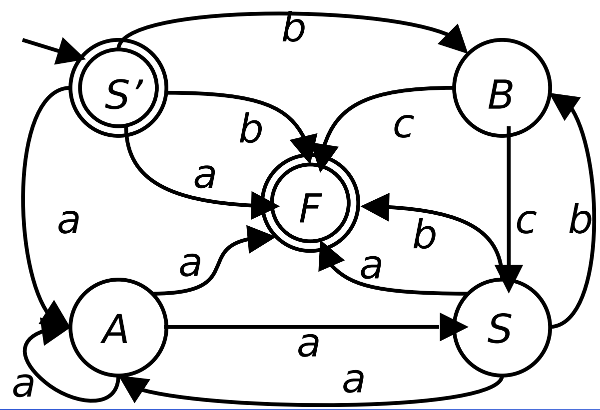
4) If the production $S' \rightarrow \varepsilon$ is in G',
   - set the state representing S' to be final state;

5) For every production $A \rightarrow aB$ in G',
   - add a transition from state A to state B labelled with a;

6) For every production $A \rightarrow a$ in G',
   - add a transition from A to the final state F.

Example:

- G:
  - $S \rightarrow a|aA|bB|\varepsilon$
  - $A \rightarrow aA|aS$
  - $B \rightarrow cS|\varepsilon$
- G':
  - $S' \rightarrow a|b|aA|bB|\varepsilon$
  - $S \rightarrow a|b|aA|bB$
  - $A \rightarrow a|aA|aS$
  - $B \rightarrow c|cS$



Making a Regular Grammar ε-Free

A regular grammar G is ε-free if it has no ε-productions except for $S \to \varepsilon$, where S is the start symbol, and S does not appear on the right hand side of the production rules.

Making a regular grammar G ε-free:

1) Copy all non ε-productions from G to G'.
   Let S be the start symbol in G';
2) For any non-terminal N which can become ε
   (While $\exists N : N \to \varepsilon$ is a production do),
   - copy every rule in which N appears on the right hand side both with and without N;
3) If $S \to \varepsilon$ was in the original set of rules,
   - add a new start symbol S' in G',
   - add the rule $S' \to \varepsilon$ and
   - copy all the production rules with S on the left hand side to ones with S' on the left hand side.

Example:

- G:
  - $S \to aA \mid bB \mid \varepsilon$
  - $A \to aA \mid a \mid \varepsilon$
  - $B \to bB \mid b \mid \varepsilon$
- 1)
  - $S \to aA \mid bB$
  - $A \to aA \mid a$
  - $B \to bB \mid b$
- 2)
  - $S \to aA \mid bB \mid a \mid b$
  - $A \to aA \mid a$
  - $B \to bB \mid b$
- 3)
  - $S' \to aA \mid bB \mid a \mid b \mid \varepsilon$
  - $S \to aA \mid bB \mid a \mid b$
  - $A \to aA \mid a$
  - $B \to bB \mid b$

Transforming FA to Regular Grammar

Transforming FA A to a regular grammar G:

1) Let T be the terminal alphabet of the grammar G – the same as that of A.

2) The set of non-terminals in G is set to be Q – the set of states of A.

3) The start state S of A will be the start symbol of G be.

4) Initially, let the set of rules in G be $\emptyset$
   For every transition $(q,a)\to q'$ of A,
   a) add the production $q\to aq'$;
   b) if q' is a final state also add the production $q\to a$.

5) If S is a final state of A add the production rule $S\to\varepsilon$.

6) If grammar is not ε-free, make it ε-free (see previous slide).

Example:

- $A=(Q,T,\delta,S,\{S,C\})$ with δ:
  - $(S,a)\to A$,
  - $(S,b)\to B$,
  - $(A,a)\to B$,
  - $(A,a)\to C$,
  - $(B,b)\to A$,
  - $(B,b)\to C$,
  - $(C,c)\to C$.
- 4)
  - $S\to aA \mid bB$,
  - $A\to aB \mid aC \mid a$,
  - $B\to bA \mid bC \mid b$,
  - $C\to cC \mid c$.
- 5)
  - $S\to aA \mid bB \mid \varepsilon$
  - $A\to aB \mid aC \mid a$
  - $B\to bA \mid bC \mid b$
  - $C\to cC \mid c$.

Transforming FA to Regular Expression

Idea: Assume, states of FA A are enumerated: $1,\ldots,n$, start state: 1.
We compute regular expressions $T(i,j,k)$ that describe all strings that take us from state i to j through states $\{1,2,\ldots,k\}$.
The language $L(A)$ is the union of all strings that take us from state 1 to a final state $f \in F$ through any state:
$L(A) = \bigcup_{f\in F} T(1,f,n)$.

Calculating $T(i,j,k)$

1) Base case, $k=0$:

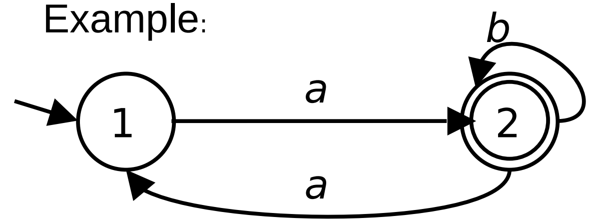
a) If $i=j$: $T(i,i,0) = \varepsilon+a+\ldots+z$, where a to z are the labels on transition arcs going from state i to itself.
If no such arcs exist, $T(i,i,0) = \varepsilon$.

b) If $i\neq j$: $T(i,j,0) = a+\ldots+z$, where a to z are the labels on transition arcs going from state i to state j
If no such arcs exist, $T(i,j,0) = \emptyset$.

2) Inductive case, $k > 0$:

$$T(i,j,k) = T(i,j,k\text{-}1) + T(i,k,k\text{-}1)(T(k,k,k\text{-}1)^*T(k,j,k\text{-}1)$$

Example:



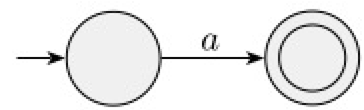
- $T(1,1,0) = \varepsilon$
  $T(2,2,0) = \varepsilon+b$
  $T(1,2,0) = a$
  $T(2,1,0) = a$
- $T(1,1,1) = \varepsilon+\varepsilon(\varepsilon)^*\varepsilon = \varepsilon$
  $T(2,2,1) = \varepsilon+b+a(\varepsilon)^*a = \varepsilon+b+aa$
  $T(1,2,1) = a+\varepsilon(\varepsilon)^*a = a$
  $T(2,1,1) = a+a(\varepsilon)^*\varepsilon = a$
- $T(1,1,2) = \ldots$
  $T(2,2,2) = \ldots$
  $\underline{T(1,2,2)} =$
  $a+a(\varepsilon+b+aa)^*(\varepsilon+b+aa) =$
  $a+a(\varepsilon+b+aa)^+ =$
  $a+a(b+aa)^* =$
  $\underline{a(b+aa)^*}$
  $T(2,1,2) = \ldots$

Transforming Regular Expression into an NFA

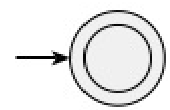
Transforming Regular Expression R into a NFA N:

NFA recognizes $L(R)$

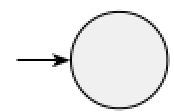
1. If $R = a$, for $a \in T$, then $L(R) = \{a\}$



2. If $R = \varepsilon$, then $L(R) = \{\varepsilon\}$



3. If $R = \emptyset$. Then $L(R) = \emptyset$



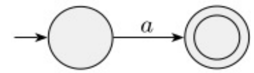
4. $R = R_1 \cup R_2$

5. $R = R_1 \cdot R_2$

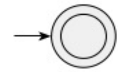
6. $R = R_1{}^*$

Transforming Regular Expression into an NFA

1. If $R = a$, for $a \in T$, then $L(R) = \{a\}$



2. If $R = \varepsilon$, then $L(R) = \{\varepsilon\}$



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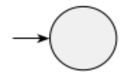
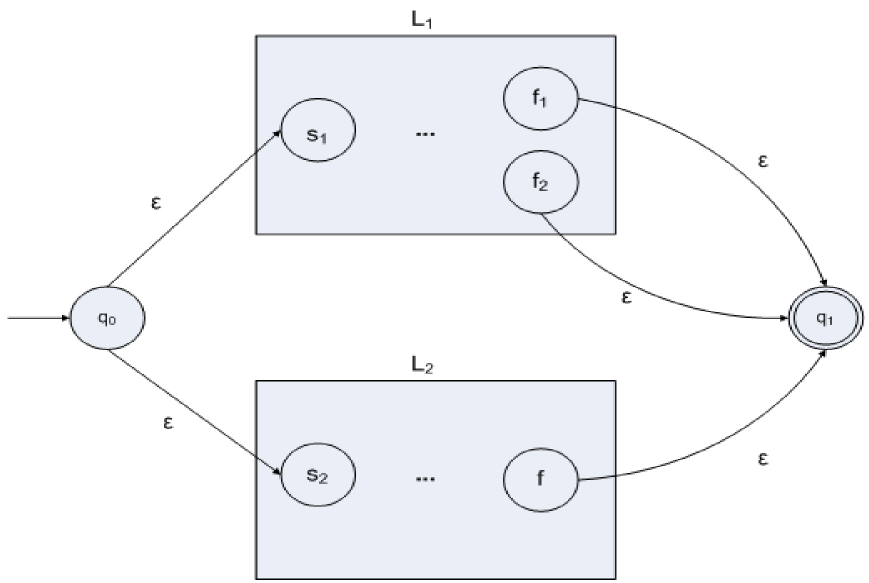


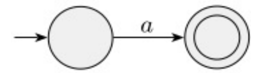
4. $R = R_1 \cup R_2$



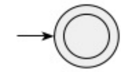
5. $R = R_1 \cdot R_2$
6. $R = R_1{*}$

Transforming Regular Expression into an NFA

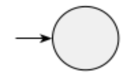
1. If $R = a$, for $a \in T$, then $L(R) = \{a\}$



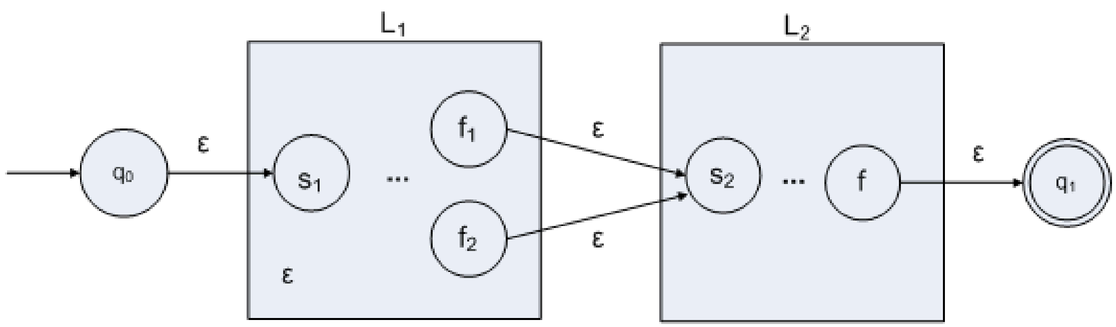
2. If $R = \varepsilon$, then $L(R) = \{\varepsilon\}$



3. If $R = \emptyset$. Then $L(R) = \emptyset$



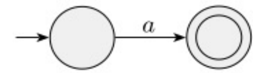
4. $R = R_1 \cup R_2$
5. $R = R_1 \cdot R_2$



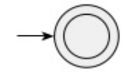
6. $R = R_1*$

Transforming Regular Expression into an NFA

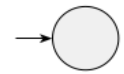
1. If $R = a$, for $a \in T$, then $L(R) = \{a\}$



2. If $R = \varepsilon$, then $L(R) = \{\varepsilon\}$



3. If $R = \emptyset$. Then $L(R) = \emptyset$



4. $R = R_1 \cup R_2$
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6. $R = R_1*$