Models of Computation

6: Probabilistic automata, Pushdown automata, Context-free languages

Probabilistic automaton

- Let $S=\{s_1, . . . ,s_n\}$ be the set of states of the **probabilistic automaton** *PA*. Reading an input symbol x in state s the automaton *PA* goes to state s_i with probability $p_i(s,x)$, where for every s and x:

$$\sum_{i=1}^{n} p_i(s,x) = 1, \quad \text{and} \quad p_i(s,x) \geq 0, i = 1, ..., n.$$

- Instead of the initial state, there is a **distribution of initial states**, i.e. every state is an initial state with a fixed probability.
- The **accepted language** $L(PA, S_f, \eta)$ depends on
  - the **final states** S_f and
  - the **cutting point** η, $0 \leq \eta < 1$.
- The **accepted language** $L(PA, S_f, \eta)$ is the set of words, for which *PA* reaches a state in S_f with a probability greater than η.

Probabilistic automaton

- An ***n*-dimensional stochastic matrix** $(p_{ij})_{1 \leq i,j \leq n}$ is a square matrix, for which

$$1.) \quad p_{ij} \geq 0 \quad (1 \leq i, j \leq n),$$
$$2.) \quad \sum_{j=1}^{n} p_{ij} = 1 \quad (1 \leq i \leq n).$$

- An ***n*-dimensional stochastic row vector (column vector)** is an *n*-dimensional row vector (column vector) whose components are are non-negative and the sum of the components is 1.
- If only one component of the stochastic row vector is 1, then it is called a **coordinate vector**.
- The *n*-dimensional unit matrix E_n is a stochastic matrix.

Probabilistic automaton

- A **finite probabilistic automaton** over an alphabet V is a triple $PA = (S, s_0, M)$, where
  - $S = \{s_1, \ldots, s_n\}$ is a finite, nonempty set of states,
  - s_0 is a n-dimensional stochastic row vector, the **distribution of the initial states**
  - M is a mapping that maps V to the set of n-dimensional stochastic matrices.
- For $x \in V$, the (i,j)-th element of the matrix $M(x)$ is $p_j(s_i,x)$, it is the probability that reading x in state s_i, PA goes to state state s_j.

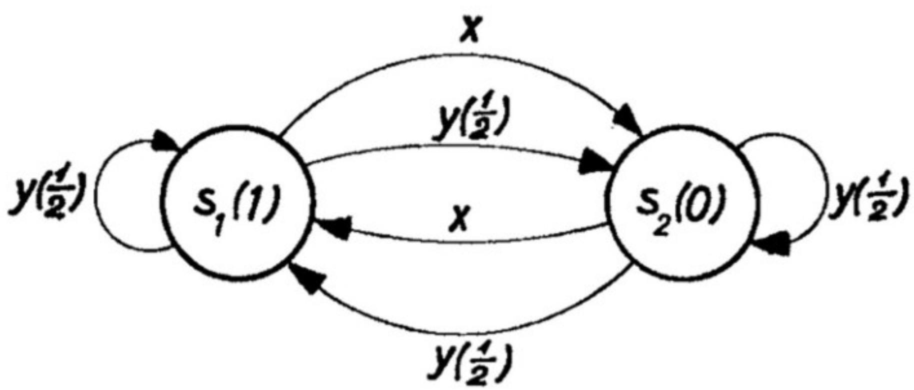
Probabilistic automaton

- Example: Consider the following probabilistic automaton: PA = ({s_1, s_2}, (1,0), M) over the alphabet {x,y}, where

$$M(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M(y) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

- The initial distribution shows that the initial state is s_1.

- The state transition digram:



Probabilistic automaton

- Let $PA = (S, s_0, M)$ be a finite probabilistic automaton over alphabet V. The function M on V can be extended to V^* as follows:
- $\hat{M}(\varepsilon) := E_n$
- $\hat{M}(x_1 \ldots x_n) := M(x_1)M(x_2)\ldots M(x_n)$, where $n \geq 2$, $x_i \in V$.
- Instead of $\hat{M}$, we write M hereafter.
- For a word $w \in V^*$, the (i,j)-th element of $M(w)$ is the probability $p_j(s_i, w)$ that processing w in state s_i the automaton PA goes to state s_j.

Probabilistic automaton

- Let $PA = (S, s_0, M)$ be a finite probabilistic automaton over an alphabet V, and $w \in V^*$. The stochastic row vector $s_0M(w)$, denoted by $PA(w)$, is the **state distribution resulting from *w***.
- Note: $PA(\varepsilon) = s_0$.

Probabilistic automaton

- Let $PA = (S, s_0, M)$ be a finite probabilistic automaton over an alphabet V, $0 \le \eta < 1$, and $\bar{s}_f$ an n-dimensional column vector, s.t. all elements of $\bar{s}_f$ are either 0 or 1.
  ($\bar{s}_f$ can be understood as a **membership function** for the final states S_f, $S_f \subseteq S$.)
- The **language accepted by *PA* with cut point η** is:
  $L(PA, \bar{s}_f, \eta) = \{ w \in V^* \mid s_0 M(w) \bar{s}_f > \eta \}$.

- A language L is called **η-stochastic** if ∃ probabilistic finite automaton $PA = (S, s_0, M)$ and column vector $\bar{s}_f$, s.t. $L = L(PA, \bar{s}_f, \eta)$ holds.
- A language L is called stochastic if it is η-stochastic for a $0 \le \eta < 1$.

Probabilistic automaton

- Example: Let $PA = (\{s_1, s_2\}, (1,0), M)$ over the alphabet $\{x,y\}$ with

$$M(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M(y) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

- Then
  - $PA(x^n) = (1, 0)M(x^n) = (1, 0)$, if n is even,
  - $PA(x^n) = (0, 1)$, if n is odd, and
  - $PA(w) = (1/2, 1/2)$ if w contains at least one y.
- Thus, for $\overline{s}_f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$L(PA, \overline{s}_f, \eta) = \begin{cases} V^* - (xx)^* & \text{if } 0 \leq \eta < 1/2, \\ x(xx)^* & \text{if } 1/2 \leq \eta < 1. \end{cases}$$

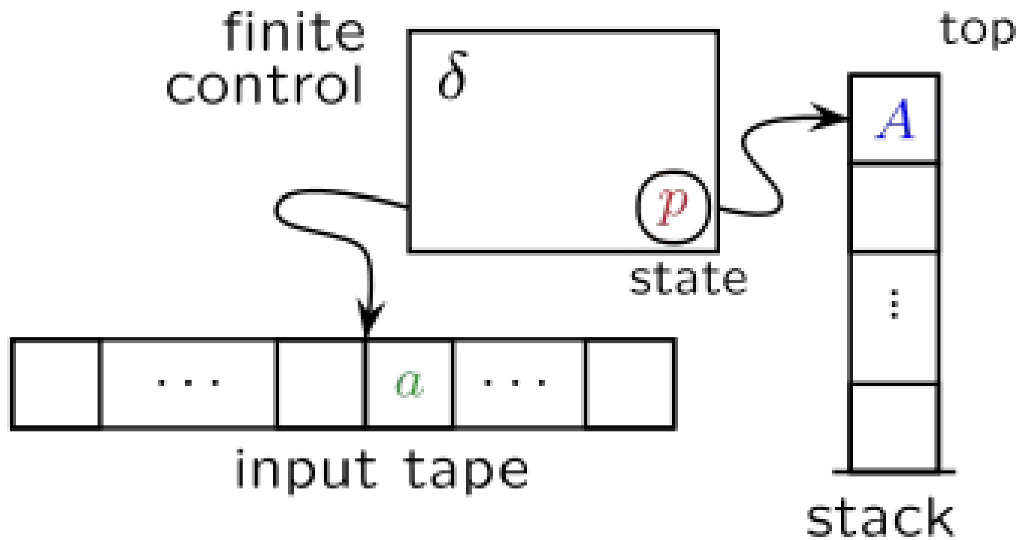
- Thus, $V^* - (xx)^*$ is, e.g., a 1/3-stochastic language, while $x(xx)^*$ is, e.g, a 2/3-stochastic language. Therfore, both are stochastic languages.

Regular and (*η*-)stochastic languages

- **Theorem 1** [Rabin 1963]: All regular languages are stochastic, but not all stochastic language is regular.
- **Theorem 2** [Rabin 1963]: All 0-stochastic languages are regular.

Pushdown automaton (PDA)

- A pushdown automaton (PDA) is a generalization of a finite automaton with (potentially) infinite stack and finite control.
- The new data is added to the top of the stack, and removed in reverse order.
- The stack is a last in, first out (LIFO) data structure.



Pushdown automata

- A **pushdown automaton (PDA)** is a 7-tuple $A = (Z, Q, T, \delta, z_0, q_0, F)$, where
  - Z is a finite set of **stack symbols** (stack alphabet),
  - Q is a finite set of **states**,
  - T is the finite set of **input symbols** (input alphabet),
  - $\delta : Z \times Q \times (T \cup \{\varepsilon\}) \to P(Z^* \times Q)$ is the **transition function**,
    - where $P(X)$ is set of finite subsets of X. (example: $\delta(z,q,a) = \{(z',q'), (z'',q'')\}$, note: **non-deterministic by default**).
  - $z_0 \in Z$ is the **initial stack symbol**,
  - $q_0 \in Q$ is the **initial state**,
  - $F \subseteq Q$ is the set of **accepting states** or **final states**.

PDA

- The symbol at the top of the stack, the current state, and the input symbol determine the transition.
- At each step, the automaton takes one element from the top of the stack (**pop**) and writes several symbols (0, 1, 2, . . .) instead (**push**).
- If $\delta(z, q, \varepsilon)$ is not empty, then so-called **ε-transition** (ε-step, ε-movement) can be performed, which allows to change the state and modify the top of the stack without reading a symbol from the input tape.
- ε-transition is possible even before reading the first input symbol or even after reading the last input symbol.

PDA

- The **configuration of the PDA** is a word of a form of *zqw*, where
  - $z \in Z^*$ is the current content of the stack,
  - $q \in Q$ is the current state, and
  - $w \in T^*$ is the unprocessed part of the input.

- *z* has its first letter at the bottom of the stack, and its last letter at the top of the stack.
- The reading head is on the first letter *w*.
- The symbol on the left of *q* is the symbol on the top of the stack and the symbol on the right of *q* is the next letter of the input to be processed.

- The initial configuration of the PDA $A=(Z,Q,T,\delta,z_0,q_0,F)$ for input $w \in T^*$ is z_0q_0w.

PDA – operations

- Let $t \in T \cup \{\varepsilon\}$, $q,r \in Q$ and $z \in Z$
  - $(\varepsilon, r) \in \delta(z, q, t)$: element z can be removed from the stack (**POP** operation)
  - $(z, r) \in \delta(z, q, t)$: the contents of the stack may remain unchanged
  - $(z', r) \in \delta(z, q, t)$: z can be replaced with z' at the top of the stack
  - $(zz', r) \in \delta(z, q, t)$: we can put z' on top of the stack (**PUSH** operation)
  - Other possibilities:
    - $(zz'z'', r) \in \delta(z, q, t)$: we can put $z'z''$ on top of the stack, z'' will be on top ($z'', z' \in Z$) .
    - In general, $(w, r) \in \delta(z, q, t)$, where $w \in Z^*$. The symbol z is replaced by the word w, s.t. the last letter of w is on the top of the stack.

PDA – reduction

- The PDA A **reduces** the configuration $\alpha \in Z^*QT^*$ to a configuration $\beta \in Z^*QT^*$ **in one step**, denoted by $\alpha \Rightarrow_A \beta$, if
  $\exists\, z \in Z$, $q,p \in Q$, $a \in T \cup \{\varepsilon\}$, $x,y \in Z^*$, and $w \in T^*$, s.t.
  $(y,p) \in \delta(z,q,a)$ and $\alpha = x\boldsymbol{zqa}w$ and $\beta = x\boldsymbol{yp}w$.

- Examples:
  - if $\delta(c,q_1,a) = \{(dd, q_2), (\varepsilon, q_4)\}$ and z_0cddcq_1 is a configuration, then
    - $z_0cdd\boldsymbol{cq_1a}babba \Rightarrow_A z_0cdd\boldsymbol{ddq_2}babba$ and
    - $z_0cdd\boldsymbol{cq_1a}babba \Rightarrow_A z_0cdd\boldsymbol{q_4}babba$ also holds.
  - if $\delta(c, q_3, \varepsilon) = \{(dd, q_2)\}$ and $z_0cddcq_3ababba$ is a configuration, then
    - $z_0cdd\boldsymbol{cq_3}ababba \Rightarrow_A z_0cdd\boldsymbol{ddq_2}ababba$
  - if $\delta(c, q_5, \varepsilon) = \emptyset$ and $\delta(c, q_5, a) = \emptyset$, then
    - $\nexists$ configuration C s.t. $z_0ccq_5aab \Rightarrow_A C$.

PDA – reduction

- The PDA A **reduces** the configuration $\alpha \in Z^*QT^*$ to a configuration $\beta \in Z^*QT^*$, denoted by $\alpha \Rightarrow^*_A \beta$, if
  - either $\alpha = \beta$,
  - or $\exists\ \alpha_1, \ldots, \alpha_n$ a finite sequence of words, s.t.
    $\alpha = \alpha_1$, $\beta = \alpha_n$ and $\alpha_i \Rightarrow_A \alpha_{i+1}$, $1 \leq i \leq n - 1$.

- The relation $\Rightarrow^*_A \subseteq Z^*QT^* \times Z^*QT^*$ is the reflexive and transitive closure of relation $\Rightarrow_A$.

- Example:
  - If $\delta(d, q_6, b) = \{(\varepsilon, q_5)\}$ and $\delta(d, q_5, \varepsilon) = \{(dd, q_2), (\varepsilon, q_4)\}$ then
    - $\#cddq_6bab \Rightarrow_A \#cdq_5ab \Rightarrow_A \#cddq_2ab$ and
    - $\#cddq_6bab \Rightarrow_A \#cdq_5ab \Rightarrow_A \#cq_4ab$.
    - So, $\#cddq_6bab \Rightarrow^*_A \#cddq_2ab$ and $\#cddq_6bab \Rightarrow^*_A \#cq_4ab$.

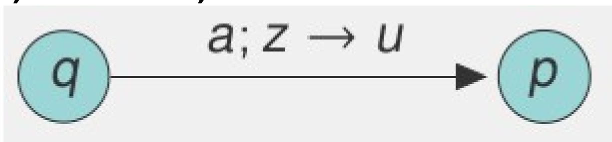
PDA – reduction

- The **accepted language with accepting state** (or with final state) by a PDA A is:

  $L(A) = \{w \in T^* \mid z_0 q_0 w \Rightarrow^*_A xp$, where $x \in Z^*, p \in F\}$.

PDA

A PDA A can be alternatively given by

- Rewriting rules
  - The set of rules is denoted by M_δ.
    Using this alternative notation:
    - $zqa \to up \in M_\delta \iff (u, p) \in \delta(z, q, a)$,
    - $zq \to up \in M_\delta \iff (u, p) \in \delta(z, q, \varepsilon)$.
    - $(p, q \in Q\ ,\ a \in T\ ,\ z \in Z\ ,\ u \in Z^*)$
- State transition diagram
  - For $p,q \in Q$, $a \in T \cup \{\varepsilon\}$, $z \in Z$, $u \in Z^*$:
    $(u, p) \in \delta(z, q, a) \iff$ 

  - Final states are indicated by double circle.
  - The start state is indicated by →.

Deterministic PDA

- The PDA $A = (Z, Q, T, \delta, z_0, q_0, F)$ is **deterministic** if for all $(z, q, a) \in Z \times Q \times T$ it holds that $|\delta(z, q, a)| + |\delta(z, q, \varepsilon)| = 1$.

- So, for all $q \in Q$ and $z \in Z$
  - either $\delta(z, q, a)$ contains exactly one element for each input symbol $a \in T$ and $\delta(z, q, \varepsilon) = \emptyset$,
  - or $\delta(z, q, \varepsilon)$ contains exactly one element and $\delta(z, q, a) = \emptyset$ for all input symbols $a \in T$.

- Remark: If for all $(z, q, a) \in Z \times Q \times T,$ it holds that $|\delta(z, q, a)| + |\delta(z, q, \varepsilon)| \leq 1$ then the PDA can be easily extended to a deterministic one accepting the same language. Thus, PDAs fulfilling this condition can be considered as deterministic in a broader sense.

Deterministic PDA

- The **acceptance (recognition) power** of deterministic PDAs is less than of non-deterministic PDAs.

- Example: Let
  - $L_1 = \{wcw^{-1} \mid w \in \{a, b\}^*\}$,
  - $L_2 = \{ww^{-1} \mid w \in \{a, b\}^*\}$.
  - L_1 can be accepted by a deterministic PDA, but L_2 not.
  - Both L_1 and L_2 can be accepted by a non-deterministic PDA.

Non-Deterministic PDA

- Example: Accepting $L_2 = \{ww^{-1} \mid w \in \{a, b\}^*\}$ non-deterministically.
  - Idea:
    - 1. read and push input symbols
      non-deterministically either repeat 1. or go to 2.
    - 2. read input symbols and pop stack sympols, compare
      if not equal reject.
    - 3. enter accept state if stack is empty.
  - Non-deterministic PDA:
    $A = (\{q_0, q_1, q_2\}, \{a, b\}, \{\$, a, b\}, \delta, q_0, \$, \{q_2\})$, where:
    - $(zt, q_0) \in \delta(z, q_0, t), \quad \forall\, t \in \{a, b\}, z \in \{\$, a, b\}$
    - $(z, q_1) \in \delta(z, q_0, \varepsilon), \quad \forall\, z \in \{\$, a, b\}$
    - $(\varepsilon, q_1) \in \delta(t, q_1, t), \quad \forall\, t \in \{a, b\}$
    - $(\$, q_2) \in \delta(\$, q_1, \varepsilon)$

Non-deterministic PDA

- Example: Accepting $L_2 = \{ww^{-1} \mid w \in \{a, b\}^*\}$ non-deterministically.
  - Idea:
    - 1. read and push input symbols
      non-deterministically either repeat 1. or go to 2.
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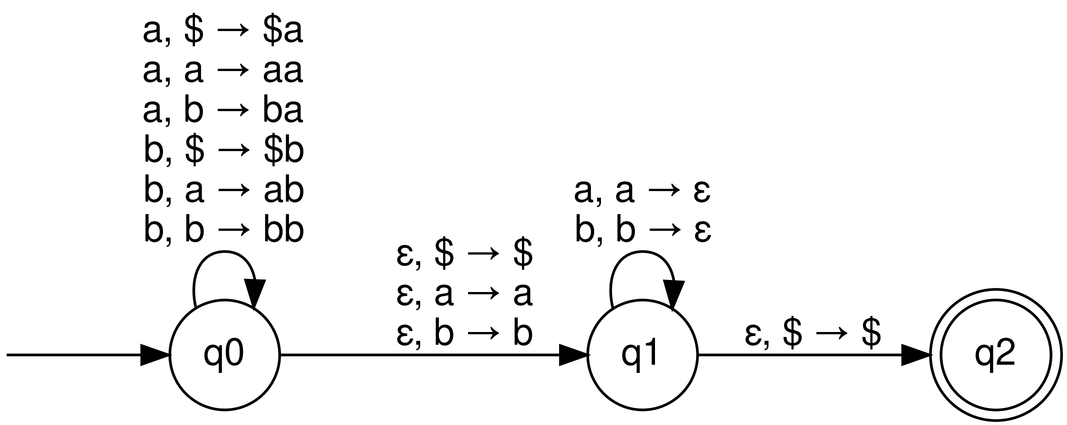


PDA

- The language **accepted** by the PDA A **with an empty stack** is
  - $N(A) = \{w \in T^* \mid z_0 q_0 w \Rightarrow^*_A p$, where $p \in Q\}$.

- Example: Let $A = (\{\$, a\}\{q_0, q_1\}, \{a, b\}, \delta, \$, q_0, \{\ \})$, where δ is:
  - $\$q_0a \rightarrow \$aq_0$
  - $aq_0a \rightarrow aaq_0$
  - $aq_0b \rightarrow q_1$
  - $aq_1b \rightarrow q_1$
  - $\$q_1 \rightarrow q_1$.

  Then $N(A) = \{a^nb^n \mid n \geq 1\}$.

PDA

- **Remark**: If the stack is empty, the operation of the automaton is blocked, since no transition is defined for the case of an empty stack. (This is why we need the symbol z_0 in the initial configuration. The set of accepting states is irrelevant to $N(A)$.)

Computing power of PDAs

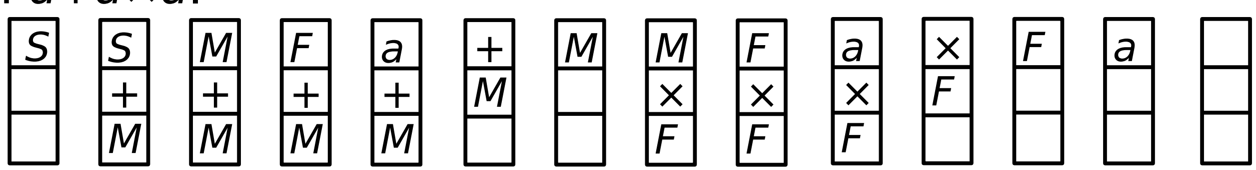
- **Theorem 3**: For every PDA A, a PDA A' can be constructed, s.t. $N(A') = L(A)$ is fulfilled.
- **Theorem 4**: For every context-free grammar G, a PDA A can be constructed, s.t. $L(A) = L(G)$.
- **Theorem 5**: For every PDA A, a context-free grammar G can be given, s.t. $L(G)=N(A)$
- Therefore, the computing power of PDAs (either we consider acceptance with accepting end state or acceptance with an empty stack) equal to the computing power of context-free (type 2) grammars.

Converting CFGs to PDAs

- **Theorem 4**: For every context-free grammar (CFG) G, a PDA A can be constructed, s.t. $L(A) = L(G)$.

- **Proof construction**: Convert the CFG G to the following PDA.
  - Push the start symbol on the stack.
  - If the top of stack is
    - Non-terminal: replace with right hand side of rule (non-deterministic choice).
    - Terminal: pop it and match with next input symbol.
  - If the stack is empty, accept.

- Example: Let $G=(N,T,P,S)$ be the CFG with $T = \{a,+,\times,(,)\}$, $N = \{S,M,F\}$, and $P=\{S \rightarrow S+M \mid M,\ M \rightarrow M\times F \mid F,\ F \rightarrow (S) \mid a\}$.
  Input: $a+a\times a$.



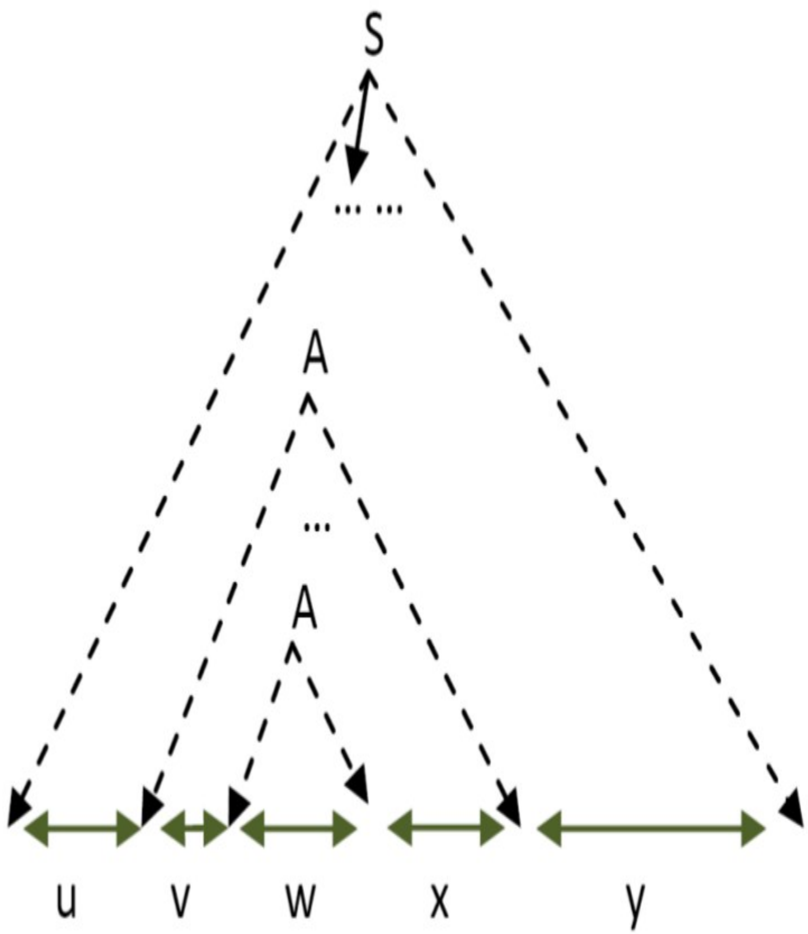
Bar-Hillel Lemma – pumping lemma for context-free languages

- A necessary condition that a language is context-free (thus, it can be recognized by a PDA).

- **Theorem 6** (Bar-Hillel lemma, or pumping lemma for context-free languages):
  For every context-free language L, there exists a natural number n, s.t. for every word $z \in L$
  with $|z|>n$, holds that z can be written as $z=uvwxy$
  ($u,v,w,x,y \in T^*$), satisfying the following 3 conditions:
  1. $|vwx| \leq n$,
  2. $vx \neq \varepsilon$,
  3. $uv^iwx^iy \in L$, for all $i \geq 0$.

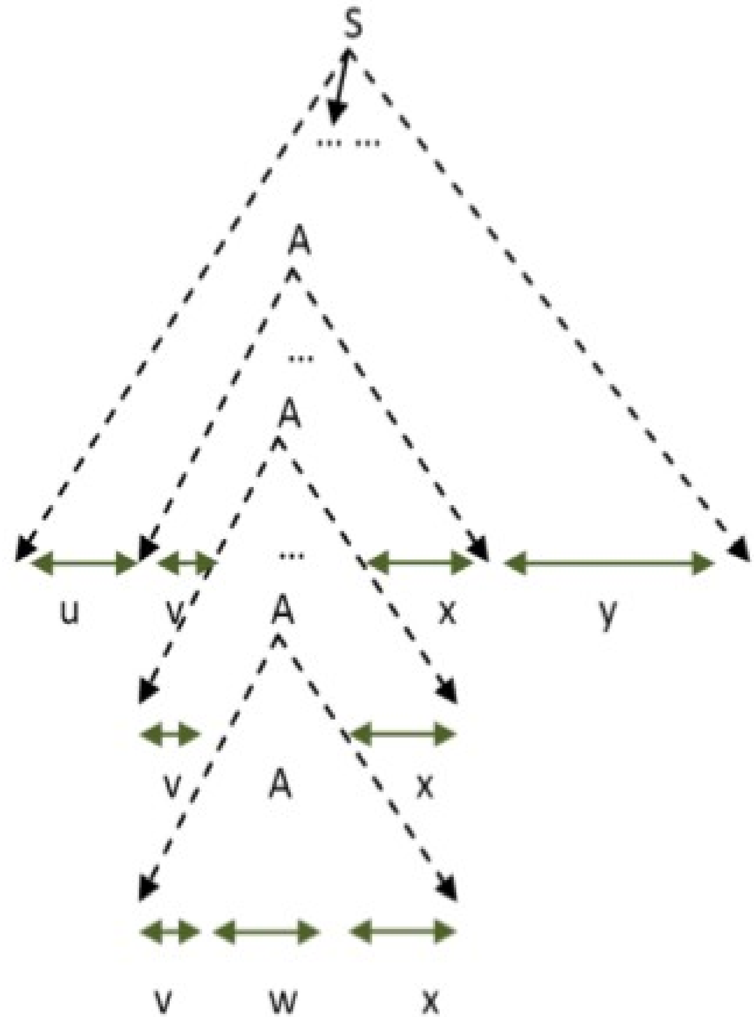
Bar-Hillel Lemma

- **Proof**: Assume, that the grammar is ε-free and given in Chomsky normal form (i.e. all production rules are of the form: $A \to BC$, or $A \to a$, or $S \to \varepsilon$).
  The derivation of a word $z \in L(G)$ can be represented by a tree T_S.
  If the depth of T_S (lengt of the longest path from S to a leaf) is k, then $|z| \leq 2^k$, due to the Chomsky normal form.
  Let N be the set of non-terminals in G and $j=|N|$. Let $n=2^{j+1}$.
  If $z \in L$ and $|z| > n$, then the longest path in the derivation tree of $S \Rightarrow^* z$ must be longer than j. Consider the last section of this path of length $j+1$. There must be a non-terminal $A \in N$ that occurs at least twice in this section.



Bar-Hillel Lemma

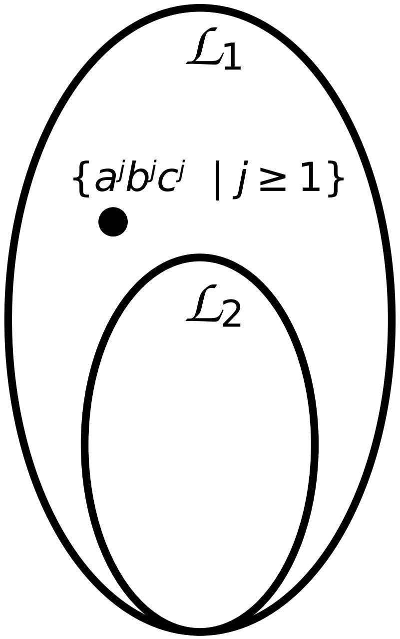
- **Proof** (cont.): Consider two such occurrences of *A* on this path. Let *r* be the word corresponding to the subtree of the first one (closer to *S*), and let *w* be the word corresponding to the other one. Then, $A \Rightarrow^* r$ and $A \Rightarrow^* w$, and *w* is a subword of *r*, so $r=vwx$ for some $v,x \in T^*$. Furthermore, $z=ury$, for some $u,y \in T^*$. Due to the choice of the occurrences of A, $|r| \leq 2^{j+1}$. On the other hand, $S \Rightarrow^* uAy$ and $A \Rightarrow^* vAx$. Therefore, $S \Rightarrow^* uv^iwx^iy$, for any $i \geq 0$. Thus, $A \Rightarrow^* vAx$ contains at least one step, and the first step must be the application of a rule of the form $A \rightarrow BC$. Therefore $|vx| \geq 1$, since *G* is ε-free. □



Application of the Bar-Hillel Lemma

- **Claim**: The language $L=\{a^j b^j c^j : j \geq 1\}$ is not context-free.

- **Proof**: Assume for contradiction, that G is a context-free grammar generating L.
  Then, by the lemma, $\exists\, n \geq 0$ s.t. $\forall$ word $z \in L$, $|z|>n$ can be written in the form $z=uvwxy$, satisfying $|vwx| \leq n$, $vx \neq \varepsilon$, and for all $i \geq 0$, $uv^iwx^iy \in L$.
  Consider a word $a^m b^m c^m$ with $m>n$.
  Since $|vwx| \leq n$, vwx can not contain all three symbols of a,b,c.
  Assume, w.l.o.g., it contains at least one a and does not contain any c. Then by pumping, for $i \geq 2$, uv^iwx^iy contains more a's than c's.
  Consequently, $uv^iwx^iy \notin L$. □



Example

- Example: A context sensitive grammar generating $L = \{a^j b^j c^j : j \geq 1\}$:

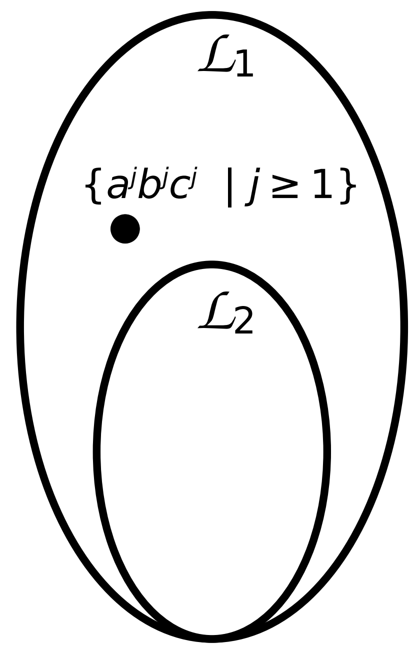
$S \rightarrow abc \mid aAbc$

$Ab \rightarrow bA$

$Ac \rightarrow Bbcc$

$bB \rightarrow Bb$

$aB \rightarrow aa \mid aaA$



Properties of CF Languages

Theorem 7: Let L and L' be CF languages. Then the languages $L \cup L'$, LL', and L^* are also CF (i.e. $\mathcal{L}_2$ is closed for the regular operations: union, concatenation, Kleene-star).

Proof: We just prove that $L \cup L'$ is CF.

- Let $G = (N,T,P,S)$ and $G' = (N',T',P',S')$ be CF gammars, s.t. $L(G) = L$ and $L(G') = L'$.
- Assume that $N \cap N' = \emptyset$. (Otherwise, rename the non-terminals in N'.)
- Let S'' be a new symbol, $S'' \notin N \cup N' \cup T \cup T'$. S'' will be the new start symbol.
- Let $G'' = (N \cup N' \cup \{S''\}, T \cup T', P'', S'')$, where $P'' = P \cup P' \cup \{S'' \rightarrow S, S'' \rightarrow S'\}$.
- Then G'' is a CF grammar and $L(G'') = L(G) \cup L(G') = L \cup L'$.
    - The derivation must begin with one of the rules $S'' \rightarrow S$ or $S'' \rightarrow S'$.
    - If it begins with $S'' \rightarrow S$, then only the rules of G can be applied.
    - If it begins with $S'' \rightarrow S'$, then only the rules of G' can be applied. $\square$

Properties of CF Languages

Theorem 8: The intersection of two CF languages is not necessarily a CF language (i.e. $\mathcal{L}_2$ is not closed for the intersection operation).

Proof: Consider the following languages over $T=\{a,b,c\}$:

- $L=\{a^n b^n c^m \mid n \geq 1, m \geq 1\}$,
- $L'=\{a^n b^m c^m \mid n \geq 1, m \geq 1\}$.
- Then $L \cap L'=\{a^n b^n c^n \mid n \geq 1\}$.
- We know from the previous slides that $L \cap L'$ is not CF.
- However, L and L' are both CF. L is generated by the CF grammar:
    - $S \rightarrow TC$
    - $C \rightarrow cC \mid c$
    - $T \rightarrow aTb \mid ab$
- L' is generated by a similar CF grammar. □

Properties of CF Languages

Theorem 9 The complement of a CF language is not necessarily CF. (i.e. $\mathcal{L}_2$ is not closed for the complement operation).

Proof:

- Assume that the complement of every CF language is CF.
- Let L and L' be two CF languages over the alphabet T.
- By the assumption, $\overline{L}$ and $\overline{L'}$ are CF.
- By Theorem 7, $\overline{L} \cup \overline{L'}$ is CF.
- Applying the assumption again, we have that $\overline{\overline{L} \cup \overline{L'}}$ is CF.
- But $\overline{\overline{L} \cup \overline{L'}} = L \cap L'$, which by Theorem 8, is not necessarily CF.
- Consequently, the assumption cannot be true. $\square$

References

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