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# Partitioned Neighborhood Spanners of Minimal Outdegree[1]

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**Abstract**

A geometric spanner with vertex set $P \subset \mathbb{R}^D$ is a sparse approximation of the complete Euclidean graph determined by $P$. We introduce the notion of *partitioned neighborhood graphs* (PNGs), unifying and generalizing most constructions of spanners treated in literature. Two important parameters characterizing their properties are the outdegree $k \in \mathbb{N}$ and the stretch factor $f > 1$ describing the 'quality' of approximation. PNGs have been thuroughly investigated with respect to small values of $f$. We, on the other hand, present in this work results about feasable values of $k$ — so to say the other extreme. The aim of minimizing this parameter rather than the first one arises from two observations:

a) It determines the amount of space required for storing PNGs.

b) Many algorithms employing a (previously constructed) spanner have running times depending on its outdegree.

Our results include, for fixed dimensions $D$ as well as asymptotically, upper and lower bounds on this optimal value of $k$. The upper bounds are constructive and yield efficient algorithms for actually computing the corresponding graphs.

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# 1 Motivation

Spanners allow for an efficient solution to many geometric problems. For given finite set $P \subset \mathbb{R}^D$, such a graph $G = (P, E)$ approximates the complete Euclidean graph up to some factor $f > 1$. $f$-spanners enabled Rao and Smith to construct a FPTAS (Fully Polynomial Time Approximation Scheme) for the Euclidean Travelling Salesperson Problem [21]. Among other applications are closest point queries [25], motion planning [7], min-cost perfect matching [27] as well as many range searching problems [1, 18].

For example, the objective of a *circular range query* is to report all those points $p$ of $P$ lying within a circle of given radius $r$ and center $c$. Having constructed an $f$-spanner $G$ for $P \subset \mathbb{R}^D$ of outdegree $k$, queries with centers $c \in P$ can be answered in nearly output sensitive running time, i.e., $\mathcal{O}(fkm)$ independent of $n = |P|$ and $m$ close to the number of points reported [14].
More precisely, this kind of geometric searching problems occurring in interactive virtual reality animations requires $\mathcal{G}$ to have only a *weakened* spanning property: The 'radius' of a path from $s$ to $t$, rather than its total length, needs being bounded by a factor $f^*$. In particular, every (strong) $f$-spanner is a weak $f^*$-spanner for some $f^*$ at most as large as $f$, usually substantially smaller.

Spanners for given $P$ are most frequently computed by an obvious generalization of *proximity graphs* [17]: Partition space $\mathbb{R}^D$ into $k \in \mathbb{N}$ convex cones $C_0, \ldots, C_{k-1}$. Then, from vertex $p \in P$, draw directed edges (*arcs*) to the closest point $u_j$ of $(p + C_j) \cap P$; do this for $j = 0 \ldots k-1$. The resulting graph is called a *partitioned neighborhood graph* (PNG). Its properties strongly depend on the number and shape of the cones $\{C_0, \ldots, C_{k-1}\} =: \mathcal{C}$ but also on the norm $\|\cdot\|$ inducing the (not necessarily Euclidean) notion of 'closest': For disadvantageous choices, this graph may be no spanner at all. However, every PNG $G = (P, E)$ is sparse with $|E| \leq kn = \mathcal{O}(n)$, $n := |P|$ and benefits from the simple construction principle, numerical robustness [2], fast computability in optimal [6] time $\mathcal{O}(n \cdot \operatorname{polylog} n)$, and locality properties that allow for incremental dynamic updates [14].

Given $D$ and small $\varepsilon > 0$, sufficient conditions on $\mathcal{C}$ and $\|\cdot\|$ have been investigated in the literature to ensure that, for any pointset $P$, the according PNG is an $(1+\varepsilon)$-spanner, i.e., an approximation of this order. Indeed, many applications — like the TSP-FPTAS mentioned above — rely on $\varepsilon \to 0$.

On the other hand, there exist cases where the outdegree $k = |\mathcal{C}|$ of $G$ is of equal importance as its stretch factor: Consider the already mentioned range query with running time proportional to $f \cdot k$. But even for algorithms that do not depend on $k$, when it comes to actually implementing it, a small outdegree may be more crucial than a small $f$.
Suppose for example that you can choose between a spanner $G_{f,k}$ of small $f$ but high outdegree $k$ and one of small outdegree $\tilde{k}$ but big stretch factor $\tilde{f}$, $G_{\tilde{f},\tilde{k}}$. The algorithm will run faster with $G_{f,k}$ but this graph requires more memory and access to secondary storage (e.g. a harddisk) being about 1000 times slower. $G_{\tilde{f},\tilde{k}}$ on the other hand entirely fits into your computer's main memory so that overall it still outperforms $G_{f,k}$ even if $\tilde{f}$ is 500 times bigger than $f$!

We therefore aim to determine the minimal value of $k$ (together with its dependence on dimension $D$) such that PNGs of this outdgree still are spanners/weak spanners. Upper bounds on this extremal problem in combinatoral geometry are not only of theoretical interest but also yield efficient algorithms for constructing such graphs.

# 2 Definitions

## 2.1 Spanners

Fix dimension $D \in \mathbb{N}$ and some norm $|\cdot|$ on $\mathbb{R}^D$. Given a path $s = p_0 \rightsquigarrow p_1 \rightsquigarrow \ldots \rightsquigarrow p_m = t$ from $s \in P$ to $t \in P \subset \mathbb{R}^D$ in some geometric graph $G = (P, E)$, the numbers

$$f(p_0, \ldots, p_m) \;:=\; \sum_{i=1}^{m} |\, p_{i-1} - p_i \,| \;/\; |\, p_0 - p_m \,| \tag{1}$$

$$f^*(p_0, \ldots, p_m) \;:=\; \max_{i=1..m} |\, p_0 - p_i \,| \;/\; |\, p_0 - p_m \,| \tag{2}$$

are called its *stretch factor* and *weak stretch factor*, respectively. An *$f$-spanner* for $P$ is a graph which for all $s, t \in P$ contains a path from $s$ to $t$ of stretch factor at most $f$; Similarly for a *weak $f^*$-spanner*...

Mind that *every* (strongly) connected graph trivially comprises an $f$-spanner for some $f < \infty$, simply by finiteness of $P$. But of course, the goal is to construct $f$-spanners with $f$ independent of $P$. Therefore, define graphs forming a family $\mathcal{G} = \{G(P) : P \subset \mathbb{R}^D \text{ finite}\}$ to be *uniform* $f$-spanners if each $G(P)$ is a $f$-spanner; and call them *uniform spanners* if $f < \infty$ exists such that they are uniform $f$-spanners. Corresponding notions will be used for weak spanners.

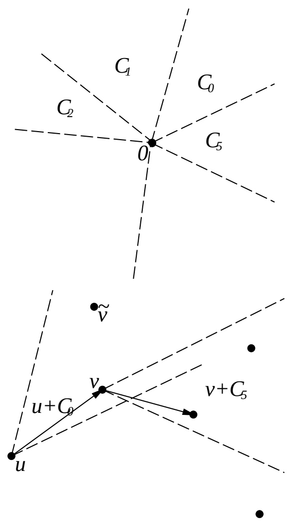
Let us remark that, by topological equivalence of any two norms on $\mathbb{R}^D$ (Claim 5.5), transition from $|\cdot|$ to $\widetilde{|\cdot|}$ affects $f$ by merely a constant factor. In particular, the notion of 'uniform spanners' does not depend on the chosen norm.

## 2.2 Partitioned Neighborhood Graphs

To formalize PNGs, consider some family $\mathcal{C} = \{C_0, C_1, .., C_{k-1}\}$ of convex cones forming a *partition* of $\mathbb{R}^D$ in the sense that it covers the whole space and is 'almost' disjoint:

$$\bigcup_{j=0}^{k-1} C_j \;=\; \mathbb{R}^D, \qquad C_j \cap C_i \;\subset\; \{0\} \quad \forall i \neq j.$$

In this context, $C \subset \mathbb{R}^D$ is said to be a *convex cone* if $\lambda(u + v) \in C$ for all $u, v \in C$ and $\lambda > 0$. Accordingly, we need a family $\mathcal{D} = \{d_0, d_1, \ldots, d_{k-1}\}$ of $k$ norms $d_j$. Then, for finite $P \subset \mathbb{R}^D$, the partitioned neighborhood graph $G(\mathcal{C}, \mathcal{D}; P) = (P, E)$ is defined by choosing, to each vertex $u \in P$ and each $0 \leq j < k$, one neighbor $v$ in $(C_j + u) \cap P \setminus \{u\} =: P_j(u)$ nearest to $u$ with respect to $d_j$.



More precisely, the edges $E$ of $G(\mathcal{C},\tilde{\mathcal{D}};P)$ are defined by three conditions:

$$\forall u \in P \quad \forall j: \quad P_j(u) = \emptyset \;\vee\; \exists v \in P_j(u): \; (u,v) \in E \tag{3}$$

$$(u,v),(u,w) \in E, v \neq w \quad \Longrightarrow \quad \forall j: \quad v \notin P_j(u) \;\vee\; w \notin P_j(u) \tag{4}$$

$$(u,v) \in E \quad \Longrightarrow \quad \exists j: \; v \in P_j(u) \;\wedge\; \forall \tilde{v} \in P_j(u): \; d_j(v-u) \leq d_j(\tilde{v}-u) \tag{5}$$

To define the *greedy path* from $s$ to $t$ in PNG $G = (P,E)$, consider the unique $C_j \in \mathcal{C}$ such that $t \in s + C_j$. Then, since $t \in P_j(s) \neq \emptyset$, there exists (3) at least and (4) at most one $v \in P_j(s)$ such that $(s,v) \in E$. Take $s \rightsquigarrow v$ as the first step and repeat from $v$ to $t$.

### 2.3 Measures of Distance

In the previous paragraph, proximity of two points was gauged with respect to some norm $d_j$. But in fact, our considerations do not rely on its symmetry property. $d_j$ may therefore be a more general distance function $d: \mathbb{R}^D \to [0,\infty) \subset \mathbb{R}$ which is

$$\begin{array}{lll} \text{positively linear} & d(\lambda v) = \lambda d(v) & v \in \mathbb{R}^D,\ \lambda \geq 0 \\ \text{nondegenerate} & d(v) \neq 0 & \mathbb{R}^D \ni v \neq 0 \\ \text{and convex.} & d(u+v) \leq d(u) + d(v) & u,v \in \mathbb{R}^D \end{array} \tag{6}$$

It is well known that such mappings uniquely correspond to the compact and convex subsets $K$ of $\mathbb{R}^D$ with 0 in their interior: According to Claim 5.5, the unit sphere $\{v \in \mathbb{R}^D : d(v) \leq 1\}$ is such a set and, vice versa, $K$'s so called *Minkowsky functional* $\mu_K$ fulfills the three conditions above,

$$\mu_K(v) \quad = \quad \inf\{\mu > 0 \;:\; v/\mu \in K\} \quad = \quad \min\{\mu \geq 0 \;:\; \mu K \ni v\}.$$

For dealing with cases where two points $v, \tilde{v} \in P_j(u)$ are both closest to $u$, we permit the distance function $d_j$ to include a *rule for breaking ties*, i.e., a total (or *linear*) order $\tilde{d}_j \subset C_j \times C_j$ that extends the partial order[2] $\{(u,v) \;:\; u,v \in C_j,\; u = v \vee d_j(u) < d_j(v)\}$ induced by $d_j$ on $C_j$ in the sense that

$$\forall v,w \in C_j: \qquad d_j(v) < d_j(w) \quad \Longrightarrow \quad (v,w) \in \tilde{d}_j.$$

By the axiom of choice, such $\tilde{d}_j$ always exists [26] and will be called an *extended norm*. Equation (5) then has to be replaced by

$$(u,v) \in E \quad \Longrightarrow \quad \exists j: \; v \in P_j(u) \;\wedge\; \forall \tilde{v} \in P_j(u): \; (v-u, \tilde{v}-u) \in \tilde{d}_j \tag{7}$$

### 2.4 Our goal

So, each choice of $\mathcal{C}$ and $\tilde{\mathcal{D}} = \{\tilde{d}_j : j = 0 \ldots k-1\}$ induces a family

$$\mathcal{G}(\mathcal{C},\tilde{\mathcal{D}}) = \{G(\mathcal{C},\tilde{\mathcal{D}};P) : P \subset \mathbb{R}^D \text{ finite}\}$$

[2] $\mathcal{O} := \{(u,v) : u,v \in C_j, d_j(u) \leq d_j(v)\}$ is no order: It violates axiom "$(u,v),(v,u) \in \mathcal{O} \Rightarrow u = v$"...

of graphs with outdegree $|\mathcal{C}|$, and our goal is to determine (for different dimensions $D$) the least value of $k = |\mathcal{C}|$ such that they constitute uniform spanners and their corresponding stretch factors:

$$k(D) := \min\Big\{k \in \mathbb{N} \;\;\Big|\;\; \exists\,\mathcal{C} \text{ disjoint partition of } \mathbb{R}^D \text{ into } k \text{ convex cones} \;\; \exists\,\tilde{\mathcal{D}} \text{ collection of } k \text{ extended norms} \;\; \exists\, f < \infty \quad \forall P \subset \mathbb{R}^D \text{ finite} : G(\mathcal{C},\mathcal{D};P) \text{ is } f\text{-spanner}\Big\} \tag{8}$$

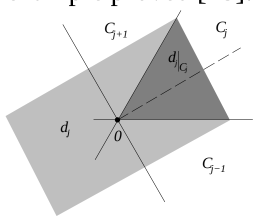
$$f(D,k) := \inf\Big\{ f > 1 \;\;\Big|\;\; \exists\,\mathcal{C} \text{ disjoint partition of } \mathbb{R}^D \text{ into } k \text{ convex cones} \;\; \exists\,\tilde{\mathcal{D}} \text{ collection of } k \text{ extended norms} \;\; \forall P \subset \mathbb{R}^D \text{ finite} : G(\mathcal{C},\mathcal{D};P) \text{ is } f\text{-spanner}\Big\} \tag{9}$$

and $k^*(D)$, $f^*(D,k)$ for uniform weak spanners, respectively.
Since any $f$-spanner is a weak $f$-spanner as well, $k^*(D) \le k(D)$ and $f^*(D,k) \le f(D,k)$ are obvious, and choosing $\tilde{k} - k$ cones empty proves $f(D,k) \ge f(D,\tilde{k})$ and $f^*(D,k) \ge f^*(D,\tilde{k})$ whenever $k \le \tilde{k}$.

# 3 Results

There already exist works which, in spite of focussing on $f \to 1$, proved specific choices for $\mathcal{C}$ and $\tilde{\mathcal{D}}$ to yield partitioned neighborhood $f$-spanners. In that way, they imply upper bounds on $k(D)$ and $f(D,k)$. Ruppert and Seidel for example proved [23]:

**3.1 Theorem:** Suppose every $C \in \mathcal{C}$ has angular diameter $\sphericalangle(C) := \sup\{\sphericalangle(a,b) : a,b \in C\}$ at most $\theta < \pi/3$. Consider (arbitrary total extension $\tilde{d_j}$ of) the norm $d_j$ with unit sphere depicted to the right. Then, for each step $p_m \rightsquigarrow p_{m+1}$ of the greedy path (see above) in $G(\mathcal{C},\tilde{\mathcal{D}};P)$ from $s = p_0$ to $t = 0$ has




$$|p_m|_2 - |p_{m+1}|_2 \ge \big(1 - 2\sin(\theta/2)\big)\cdot|p_{m+1} - p_m|_2 \tag{10}$$

Since $k = 7$ equally sized wedges do form such a partition $\mathcal{C}$ in dimension $D = 2$, $G(\mathcal{C},\tilde{\mathcal{D}};P)$ is an Euclidean $f$-spanners for

$$f(2,7) \;\le\; \frac{1}{1 - 2\sin(\theta/2)}\Big|_{\theta=2\pi/7} \;\approx\; 7.57, \qquad k(2) \le 7 \quad \square$$

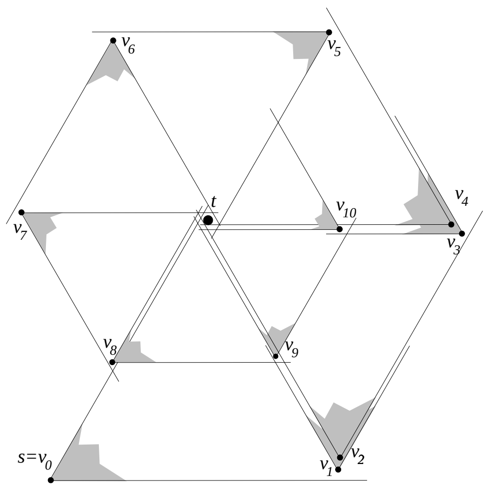
The table below shows a compilation of such results as well as the improved upper bounds presented in this paper. Mind that we are the first to prove lower bounds!

| dim | reference | bound | bound |
|---|---|---|---|
| $D=2$ | Keil, Gutwin 1991 | $k(2) \leq 9$ | $k^*(2) \leq 9$ |
| | Ruppert, Seidel 1992 | $k(2) \leq 7$<br>$f(2,7) \leq 7.57$ | $k^*(2) \leq 7$<br>$f^*(2,7) \leq 7.57$ |
| | Fischer, Meyer a.d. Heide, Strothmann '97 | | $k^*(2) \leq 6$<br>$f^*(2,6) \leq 2$ |
| | Fischer, Lukovszki, Ziegler 1998 | | $k^*(2) \leq 4$<br>$f^*(2,4) \leq 2.29$ |
| | *new* | $k^*(2) \geq 4$ | $k^*(2) = 4$ |
| | *conjecture* | $k(2) = 4$ | |
| $D=3$ | Hardin, Sloane, Smith 1994 | $k(3) \leq 20$<br>$f(3,20) \leq 88.1$ | $k^*(3) \leq 20$<br>$f^*(3,20) \leq 88.1$ |
| | *new* | | $k^*(3) \leq 8$<br>$f^*(3,8) \leq 2.53$ |
| | *new* | $k^*(3) \geq 5$ | $k^*(3) \geq 5$ |
| $D \to \infty$ | Rogers 1963 | $k(D) \leq 2^{\mathcal{O}(D)}$ | $k^*(D) \leq 2^{\mathcal{O}(D)}$ |
| | *new* | $k(D) \geq D+2$ | $k^*(D) \geq D+2$ |

Combining Theorem 3.1 with the following result from Coding Theory due to Hardin, Sloane, and Smith [16] implies $k(3) \leq 20$ and $f(3,20) \leq 88.1$:

**3.2 Theorem:** There exists a covering of the unit sphere $\mathcal{S}^3 \subset \mathbb{R}^3$ with $k = 20$ caps of angular diameter $\theta \approx 59.25°$. ☐

For $60° \leq \theta \leq 90°$, greedy paths become unboundedly long (see figure) but remain of bounded diameter. As this may include the possibility of cycling infinitely without ever reaching $t$, it does not necessarily imply obtaining a weak spanner. In fact, which of the cones' boundaries are open and which ones closed turns out a sophisticated combinatorial challenge for $\theta = 90°$; similarly does the particular extension of norm $d_j$ which is not arbitrary any longer. The planar cases have been treated in [14] and [12]:



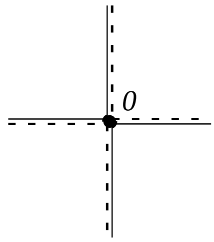

**3.3 Theorem:** Let $D = 2$, $k \geq 7$, and $\mathcal{C}$ consist of $k$ consequtive wedges

$$C_j = \left\{ (r\cos\varphi, r\sin\varphi) : r > 0, \frac{2\pi}{k} j \leq \varphi < \frac{2\pi}{k}(j+1) \right\}, \qquad j = 0 \ldots k-1.$$

Then, for $\tilde{\mathcal{D}}$ as in Theorem 3.1, $G(\mathcal{C},\tilde{\mathcal{D}};P)$ is a weak Euclidean $f^*$-spanner for

$$f^* \;\le\; \max\Big\{\sqrt{1+48\sin^4(\pi/k)},\sqrt{5-4\cos(2\pi/k)}\Big\}. \quad \square$$

**3.4 Theorem:** Let $D=2$ and $\mathcal{C}=\{C_0,C_1,C_2,C_3\}$ the four canonical quadrants with boundaries open/closed as shown to the right. Let $\tilde{\mathcal{D}}=\{\tilde{d_0},\ldots,\tilde{d_3}\}$, $\tilde{d_j}$ arbitrary total extension of




$$\big\{(v,w)\;:\; v,w\in C_j,\\ \quad (|v|_\infty<|w|_\infty) \quad\vee\quad (|v|_\infty=|w|_\infty \,\wedge\, |v|_0<|w|_0)\;\big\}$$

i.e. the lexicographical order on $C_j$ induced by $v\mapsto\big(|v|_\infty,|v|_0\big)$,

$$|v|_p=\Big(\sum_i|v_i|^p\Big)^{1/p},\qquad |v|_\infty=\max_i|v_i|,\qquad |v|_0=\min_i|v_i|$$

Then potential function $\Phi(s)=\big(|s-t|_\infty,\varphi(s-t)\big)$, $\varphi(x,y)=|x+y|$, strictly decreases in each step of the greedy path.
In particular, the latter does reach $t$ with (not necessarily strictly) decreasing $|\cdot-t|_\infty$.
So, $\mathcal{G}(\mathcal{C},\tilde{\mathcal{D}})$ are uniform weak spanners of Euclidean weak stretch

$$f^* \quad\le\quad \Big\{|a-b|_2/|a|_2 : |a|_\infty=1=|b|_\infty\Big\} \qquad=\quad \sqrt{3+\sqrt{5}}. \quad\square$$

**3.5 Theorem:** Given $P\subset\mathbb{R}^2$, $n:=\#P$, the graph $G(\mathcal{C},\tilde{\mathcal{D}};P)$ of Theorem 3.3 can be computed by sequentially performing $k$ sweep line algorithms, each of time $\mathcal{O}(n\cdot\log n)$. The graph of Theorem 3.4 can be computed in the same magnitude of time. $\square$

The first of our contributions generalizes this to 3$D$:

**3.6 Theorem:** Let $D=3$ and consider the 8 canonical octants. Turn them into a partition by including each of their common boundaries to one of them and excluding it from the others in the following way: $\mathcal{C}:=\big\{C_{\bar\iota}:\bar\iota\in\{+,-\}^3\big\}$,

$$C_{\bar\iota}:=\big\{q\in\mathbb{R}^3, q\neq 0,\, \bar f(\operatorname{sgn}q_x,\operatorname{sgn}q_y,\operatorname{sgn}q_z)=\bar\iota\big\} \tag{11}$$

for $\bar f=(f_x,f_y,f_z):\{+,0,-\}^3\to\{+,-\}^3$, given by $\bar f\big|_{\{+,-\}^3}=$identity and otherwise

| | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 0 | 0 | 0 | + | + | − | − | + | + | − | − | 0 | 0 | 0 | 0 | + | − | 0 |
| $y$ | + | + | − | − | 0 | 0 | 0 | 0 | + | − | + | − | 0 | 0 | + | − | 0 | 0 | 0 |
| $z$ | + | − | + | − | + | − | + | − | 0 | 0 | 0 | 0 | + | − | 0 | 0 | 0 | 0 | 0 |
| $f_x$ | + | + | − | − | + | + | − | − | + | + | − | − | + | − | + | − | + | − | + |
| $f_y$ | + | + | − | − | + | − | + | − | + | − | + | − | + | − | + | − | + | − | + |
| $f_z$ | + | − | + | − | + | − | + | − | + | + | − | − | + | − | + | − | + | − | + |

Furthermore, be the partial lexicographical order induced by $C_{\bar\iota}\ni v\mapsto\big(|v|_\infty,|v|_1\big)$ extended to a total one $d_{\bar\iota}$. Then, $G(\mathcal{C},\tilde{\mathcal{D}};P)$ has weak stretch factor 2 with respect to $|\cdot|_\infty$ and Euclidean weak stretch $f^*\le\sqrt{(7+\sqrt{33})/2}$. $\square$

**3.7 Theorem:** Given $P \subset \mathbb{R}^3$, $n := \#P$, the graph $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ of Theorem 3.6 can be computed in time $\mathcal{O}(n \cdot \log^2 n)$ from 48 sweep plane passes.

The lower bounds mentioned are immediate consequences of the following two results:

**3.8 Theorem:** In the planar case $D = 2$, no choice of $\mathcal{C}$ and $\tilde{\mathcal{D}}$ of size $k \leq 3$ makes $\mathcal{G}(\mathcal{C}, \tilde{\mathcal{D}})$ a family of uniform weak spanners, since there exists $P \subset \mathbb{R}^2$ and $s, t \in P$ such that no path from $s$ to $t$ is present at all. □

**3.9 Theorem:** In cases $D \geq 3$, no choice of $\mathcal{C}$ and $\tilde{\mathcal{D}}$ of size $k < D + 2$ makes $\mathcal{G}(\mathcal{C}, \tilde{\mathcal{D}})$ a family of strongly connected graphs (and thus neither of uniform weak spanners). More precisely, inequalities $k(D) \geq k(D-1) - 1$ and $k^*(D) \geq k^*(D) - 1$ hold. □

This does not rule out the possibility to obtain (weak) spanners for $D = 3$, $k = 6$. We can, however, exclude the choice of 6 convex cones arising canonicaly from the faces of a cube ($\mathring{C}$ and $\overline{C}$ denote topological interior and closed hull of $C$, respectively):

**3.10 Theorem:** Suppose $\mathcal{C} = \{\tilde{C}_{\bar{\iota}} : \bar{\iota}\}$, $\bar{\iota} = (i, s) \in \{x, y, z\} \times \{+, -\}$ and

$$\mathring{C}_{\bar{\iota}} \subset \tilde{C}_{\bar{\iota}} \subset \overline{C}_{\bar{\iota}}, \qquad C_{\bar{\iota}} := \{q \in \mathbb{R}^3, q \neq 0, |q|_\infty \leq s \times q_i\}.$$

Then to any collection $\tilde{\mathcal{D}}$ of 6 extended norms there exists $P \subset \mathbb{R}^3$ such that $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ is not strongly connected. □

## 4 Conclusions/Open Problems

We presented upper and lower bounds for the numbers $k(d)$ and $k^*(d)$, i.e., the minimally achievable outdegree such that partitioned neighborhood graphs (PNGs) still form spanners and weak spanners, respectively.

The notion of PNGs we suggested is very general since we allow for *arbitrary* partitions of space into convex cones. Furthermore, the neighbor needs not be 'nearest' in the Euclidean sense but with respect to *any* nondegenerate convex distance function $d$ which may be different for each cone and even be equipped with a rule for breaking ties between equally distant points. In particular, most existing constructions of spanners are PNGs.

We do not aim to find the minimal outdegree of arbitrary spanners (this is well known, anyway: 3. See [8]) but of those which can be constructed in nearly linear time $\mathcal{O}(n \cdot \text{polylog}\, n)$. Our upper bounds are constructive and yield practical algorithms of this optimal time complexity. We obtained lower bounds by proving that for smaller outdegree, the corresponding PNGs will in general be not only of unbounded length and diameter but even disconnected (an important observation, see below!).

This was done by a new technical tool which took care of the vast range of possible choices for the distance functions. This allowed us to reduce the topological part of the problem. The remaining challenge of considering all partitions of space into $k$ convex cones was still difficult enough: finding so called cycles, a simultaneously combinatorial and geometric property of a family of cones.

For $k^*(2) = 4$, our bounds are tight. Concerning the gap between 4 and 7 for $k(2)$, we conjecture that the actual value is 4, too. In order to prove the appropriate upper bound, greedy paths do not suffice any more.
In higher dimensions, we believe $k(3) = k^*(3) = 8$ and $k(D) = k^*(D) = \Theta\big(2^D\big)$. PNGs then would have the interesting property that

- they are are either disconnected or
- permit paths of uniformly bounded length.

The other cases

- unbounded diameter and
- bounded diameter but unbounded length

could not occur by themselves. This is different for arbitrary families of geometric graphs!

Apart from filling the remaining gaps by tightening the upper and lower bounds, another direction of research seems promising: What happens if the notion of 'closest' is not deduced from an extended norm $\tilde{d}$ but from an *arbitrary* total order $\preceq$ of cone $C$? Even in case $\preceq$ is required to be compatible with the cone's operations "$\cdot$" and "$+$" in the sense of [15], i.e.,

$$u \preceq v, \quad \lambda \geq 0, \quad w \in C \qquad \Longrightarrow \qquad \lambda \cdot u \;\preceq\; \lambda \cdot v \quad \wedge \quad u + w \;\preceq\; v + w,$$

we have no idea whether this actually affects the values $k(d)$ and $k^*(d)$ or does not.

# 5 Proofs of lower bounds

For proving a lower bound, *every* choice of $\mathcal{C}$ and $\tilde{\mathcal{D}}$ has to be taken into consideration. The following important result allows us to get at least rid off the norms:

**5.1 Definition:** Be $\mathcal{C}$ a collection of (not necessarily disjoint neither covering) convex cones $C \subset \mathbb{R}^D$. A *cycle* of $\mathcal{C}$ is a finite sequence $(c_0, c_1, \ldots, c_{L-1}, c_L = c_0)$ of nonzero points $c_l \in \mathbb{R}^D$ such that

$$\forall l = 0 \ldots L-1 \quad \exists C \in \mathcal{C}: \qquad 0 \in c_l + \mathring{C} \quad \wedge \quad c_{l+1} \in c_l + C \tag{12}$$

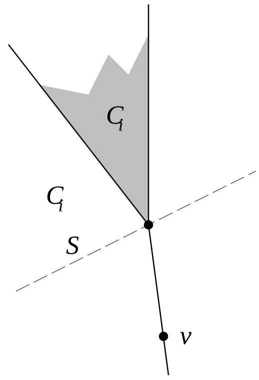
**5.2 Proposition:** Fix some partition $\mathcal{C}$ of $\mathbb{R}^D$ into convex cones. Suppose there exists subspace $S$ and nonzero vector $v \in \mathbb{R}^D$ such that

$$\mathcal{C}' := \big\{ C \cap S : C \in \mathcal{C}, v \in \overline{C} \big\} \tag{13}$$

contains a cycle. Then for *any* choice of extended norms $\tilde{\mathcal{D}}$ there exists $P \subset \mathbb{R}^D$ such that $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ is not strongly connected. Here we identify $S$ with $\mathbb{R}^{D'}$, $D' < D$. ☐

**Proof of Theorem 3.8:** In case $k = 3$,

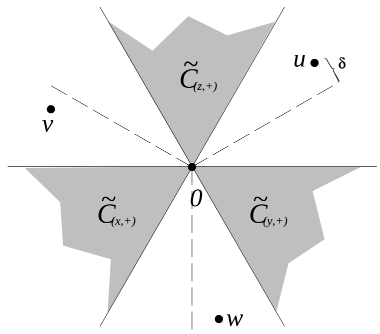
$$\sum_{i=0}^{k-1} \sphericalangle(C_i) = 360° \qquad \Longrightarrow \qquad \exists i : \sphericalangle(C_i) \leq 120° < 180°.$$




The tangent line $S$ at $C_i$ through 0 therefore intersects precisely the other two cones. Choose $v \neq 0$ from their common boundary. Identifying $S$ with $\mathbb{R}^d$, we have $\mathcal{C}' = \{(-\infty, 0], [0, +\infty)\}$ with obvious cycle $(-42, +42, -42)$.
In case $k = 2$, both cones are halfspaces. Choose $v$ from their boundary and $S$ perpendicular to $v$. Case $k = 1$ is trivial. □

**Proof of Theorem 3.10:** Consider $v = (1, 1, 1)$.
$S = \{u \in \mathbb{R}^3 : u \perp v\} \cong \mathbb{R}^2$ via vectorspace isometry

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\sqrt{1/2}x - \sqrt{1/6}y \\ +\sqrt{1/2}x - \sqrt{1/6}y \\ +\sqrt{2/3}y \end{pmatrix}$$




The collection of cones $\mathcal{C}'$ induced by $\mathcal{C}$ is shown to the right; boundaries may be open or closed. Now, let $0 < \delta < 30°$ arbitrary. Then, points $\tilde{u}, \tilde{v}, \tilde{w} \in S$,

$$\begin{aligned} \tilde{u} &= (-\sqrt{2/3}\cos\delta, +\sqrt{1/6}\cos\delta - \sqrt{1/2}\sin\delta, +\sqrt{1/6}\cos\delta + \sqrt{1/2}\sin\delta) \\ \tilde{v} &= (+\sqrt{1/6}\cos\delta + \sqrt{1/2}\sin\delta, -\sqrt{2/3}\cos\delta, +\sqrt{1/6}\cos\delta - \sqrt{1/2}\sin\delta) \\ \tilde{w} &= (+\sqrt{1/6}\cos\delta - \sqrt{1/2}\sin\delta, +\sqrt{1/2}\sin\delta + \sqrt{1/6}\cos\delta, -\sqrt{2/3}\cos\delta) \end{aligned}$$

corresponding to

$$\begin{aligned} u &= \big(\cos(\ 30° + \delta), \sin(\ 30° + \delta)\big) \\ v &= \big(\cos(150° + \delta), \sin(150° + \delta)\big) \qquad \in \mathbb{R}^2 \\ w &= \big(\cos(270° + \delta), \sin(270° + \delta)\big) \end{aligned}$$

obviously form a cycle $(u, v, w, u)$ of $\mathcal{C}'$. □

**5.3 Claim:** If $C \subset \mathbb{R}^D$ is convex, $a \in \overline{C}$, $b \in \mathring{C}$, then $a + b \in \mathring{C}$.
**Proof:** Let $a_n \in C$ be a sequence convergent to $a$. $b \in \mathring{C}$, therefore exists a ball $B$ around $b$ such that $B \subset C$. For each $\tilde{b} \in B$ and each $n$, $a_n + \tilde{b} \in C$ by convexity and, letting $n \to \infty$, $a + \tilde{b} \in \overline{C}$. This proves that the whole ball $a + B$ around $a + b$ lies within $\overline{C}$, so $a + b \in \mathring{C}$. □

**Proof: (Proposition 5.2)** Be $(c_0, c_1, \ldots, c_{L-1}, c_L = c_0)$ a cycle of $\mathcal{C}'$, i.e., $c_{l+1} \in c_l + (C_l \cap v^\perp) \subset c_l + C_l$ and $-c_l \in \mathring{C}_l$. $v \in \overline{C_l}$, therefore $t := \mu v \in \overline{C_l} \subset \mathbb{R}^D$ for any $\mu > 0$. Application of Claim 5.3 to $a = t$, $b = -c_l$ ensures $t \in c_l + C_l$; $c_{l+1} \in c_l + C_l$, anyway. Now let $\tilde{d}_l \in \tilde{\mathcal{D}}$ belong to $C_l \in \mathcal{C}$, $d_l$ the distance function which $\tilde{d}_l$ extended to. Since $d_l(c_{l+1}-c_l)$ is independent of $\mu$ and $d_l\big(t-c_l\big) \overset{(*)}{\leq} \mu d_l(v) - d_l(c_l) \to \infty$ as $\mu \to \infty$ (Claim 5.5),

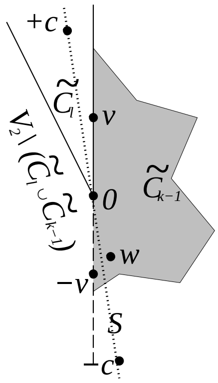
$$\exists \lambda > 0: \qquad \forall l = 0,\dots,L-1: \quad c_{l+1}, t \in c_l + C_l, \quad d_l\big(c_{l+1}-c_l\big) < d_l\big(t-c_l\big)$$

Letting $P = \{t, c_0, \dots, c_{L-1}\}$, no $c_l$ will therefore have an arc to $t$ in $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$. □

**Proof of Theorem 3.9:** Suppose $\mathcal{C} = \big\{C_0,\dots,C_{k-1}\big\}$, $\tilde{\mathcal{D}} = \big\{\tilde{d}_0,\dots,\tilde{d}_{k-1}\big\}$ for $k < D+2$. Consider the case $D=3$. Since $C_{k-1}$ is convex, we can find at boundary point 0 some tangent hyperplane $H$ not touching $C_{k-1}$ other than in 0. The intersections with and restrictions to this $(D-1)$-dimensional subspace

$$\begin{array}{llll} C_0' = C_0 \cap H & C_1' = C_1 \cap H & \dots & C_{k-2}' = C_{k-2} \cap H \\ \tilde{d}_0' = \tilde{d}_0\big|_H & \tilde{d}_1' = \tilde{d}_1\big|_H & \dots & \tilde{d}_{k-2}' = \tilde{d}_{k-2}\big|_H \end{array}$$

therefore form a partition $\mathcal{C}'$ of $H \cong \mathbb{R}^2$ into $k-1 < 4$ convex cones and a family $\tilde{\mathcal{D}}'$ of extended norms thereon. Now, take the counter example $P \subset \mathbb{R}^2$ from Theorem 3.8 and place it onto $B \subset \mathbb{R}^3$: The resulting PNGs are disconnected. $\surd$
In cases $D > 3$, employ the same argument as induction step.




Attentive readers might have remarked that in some degenerate cases, $C_{k-1}$ may include angles as large as 180° *and* be closed. Here, we cannot guarantee the tangent hyperplane to be even 'almost' disjoint to $C_{k-1}$. Fortunately, the subsequent Claim permits a characterization of these particularities! So if $H$ with the required property does not exist, take $v \in C_{k-1}$, $-v \in \overline{C_{k-1}}$. Suppose first that $-v \notin C_{k-1}$. Then, to $\varepsilon = |v|/2 > 0$ we can find $w \in C_{k-1}$ such that $|w-(-v)| < \varepsilon$ and in particular $w$ not colinear to $-v, v$. Plane $V_2 := \operatorname{span}\{-v, v, w\}$ has the property that $\tilde{C}_{k-1} := V_2 \cap C_{k-1}$ is a halfopen wedge of 180°.
The partition $\tilde{\mathcal{C}} = \big\{C \cap V_2 : C \in \mathcal{C}\big\}$ induced on $V_2$ by $\mathcal{C}$ will therefore look like to the right: Since $\tilde{C}_{k-1}$ is closed at $v$, the adjacent wedge $\tilde{C}_l \in \tilde{\mathcal{C}}$ must form a nonzero angle (whereas the one containing $-v$ could perhaps be nothing more than a ray). Once again refer to the Claim below to understand the existence of a line $S$ through 0 which does not touch $V_2 \setminus (\tilde{C}_l \cup \tilde{C}_{k-1})$. Points $c \neq 0$ and $-c$ on this line then form a cycle $(+c, -c, +c)$ of

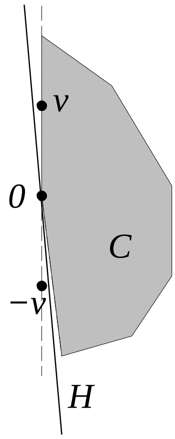
$$\mathcal{C}' = \big\{C \cap S : C \in \tilde{\mathcal{C}}, v \in \overline{C}\big\} = \big\{C \cap S : C \in \mathcal{C}, v \in \overline{C}\big\}.$$

Application of Proposition 5.2 completes this case. In case $-v \in C_{k-1}$, $S := \operatorname{span}\{v\}$ and $(+v, -v, +v)$ similarly forms a cycle. □

**5.4 Claim:** Be $C \subset V_D = \mathbb{R}^D$ convex, $p \in \partial C$. Then, there exists

- either a $(D-1)$-dimensional hyperplane $H_{D-1} \ni p$ such that $H_D \cap C \subset \{p\}$
- or $v \in V_D$ such that $p + v \in C$ but $p - v \notin \overline{C}$.

**Proof:** W.l.o.g. $p=0$ and presume $-v\notin\overline{C}\ \ \forall v\in C$. The claim that $H$ with the required properties exists is trivial for $D=1$ and obvious in dimension 2 (see sketch to the right). Proceed now by induction to $D+1$. Be $q\in C$ arbitrary. Consider some $D$-dimensional subspace $V_D\subset V_{D+1}$ containing $q$. And consider the plane $V_2$ going through $q$ and 0 perpendicular to $V_D$, i.e. $V_2\cap V_D$ is one-dimensional.




$0\notin C\cap V_2=:C_2$ is a convex set in $V_2$ with 0 at is boundary fulfilling $-v\notin\overline{C_2}\ \ \forall v\in C_2$. For this reduction to $D=2$, a 1-dimensional disjoint hyperplane $H_1$ (simply a line) through 0 is already known to exist. Now consider the projection of $C$ parallel to this line onto $V_D$,

$$\Pi\big(H_1,V_D;C\big)=\big\{L\cap V_D : L \text{ line through } c \text{ parallel to } H_1, c\in C\big\}.$$

$0\notin C_D:=\Pi\big(H_1,V_D;C\big)\subset V_D$ too is convex (since projection $\Pi\big(H_1,V_D;\cdot\big)$ linear mapping) and 0 a boundary point of $C_D$ (as $\Pi$ is continuous). Furthermore, $-\tilde{v}\notin\overline{C_D}$ $\forall\tilde{v}\in C_D$!

Indeed, be $\tilde{v}=\Pi(v)$, $v\in C$ and $-\tilde{v}=\Pi(w)\in\overline{C_D}$, $w\in C$. Be definition of $\Pi$, lines $A$ and $B$ through $v,\tilde{v}$ and $w,-\tilde{v}$, respectively, are parallel to $H_1$. $A$, $B$ and $H_1$ therefore lie on a common twodimensional subspace $\tilde{V}_2$. Line $C\subset\tilde{V}_2$ through $v,w$ however is not parallel to $H_1$ $\big($ otherwise $\Pi(v)=\Pi(w)$ $\big)$ and so intersects $H_1$ in some point $u$ which, by convexity, contains to $C$ as well. But $u\in H_1\cap C$ contradicts the choice of $H_1$ to be disjoint to $C$.

Induction hypothesis is thus applicable to $C_D$ and supplies a $(D-1)$-dimensional hyperplane $H_{D-1}$ through 0 disjoint to it.

$H_D:=H_{D-1}+H_1$ then will do the job: Suppose $c\in C\cap H_D$. Then its projection $\Pi(c)$ will be on $\Pi(C)\cap H_{D-1}$ contradiction that $H_{D-1}$ is disjoint to $C_D$. ☐

**5.5 Claim: (Topological Equivalence)** Be $d_a$ and $d_b$ nondegenerate convex distance functions on $\mathbb{R}^D$. Then there exist real numbers $0<\lambda<\Lambda<\infty$ such that

$$\forall v\in\mathbb{R}^D:\qquad \lambda\cdot d_a(v)\quad\le\quad d_b(v)\quad\le\quad\Lambda\cdot d_a(v).$$

**Proof:** Denote $e^{(i)}$ the $i$-th canonical unit vector of $\mathbb{R}^D$, i.e., $e^{(i)}_j=\delta_{ij}$ for $1\le i,j\le D$. We start with the case $d_a=|\cdot|_1 : v=\sum_i v_ie^{(i)}\mapsto\sum_i|v_i|$. Define $\Lambda:=\max_i d_b\big(\pm e^{(i)}\big)$.

$$\begin{aligned}
\text{Then}\quad d_b(v)\ &=\ d_b\Big(\sum_i v_ie^{(i)}\Big)\ =\ d_b\Big(\sum_{i:v_i>0}|v_i|(+1)e^{(i)}+\sum_{i:v_i<0}|v_i|(-1)e^{(i)}\Big)\\
&\overset{(6)}{\le}\ \sum_{i:v_i>0}d_b\big(|v_i|(+1)e^{(i)}\big)+\sum_{i:v_i<0}d_b\big(|v_i|(-1)e^{(i)}\big)\\
&=\ \sum_{i:v_i>0}|v_i|d_b\big(+e^{(i)}\big)+\sum_{i:v_i<0}|v_i|d_b\big(-e^{(i)}\big)\\
&\le\ \sum_{i:v_i>0}|v_i|\Lambda+\sum_{i:v_i<0}|v_i|\Lambda\qquad=\ \big|v\big|_1\cdot\Lambda
\end{aligned}$$

This in turn implies that $d_b$ is continuous: Let $v^{(n)}$ a sequence in $\mathbb{R}^D$ converging to $v$.

$$\begin{aligned} \text{Then} \quad & d_b\big(v^{(n)}\big) - d_b\big(v\big) \overset{(*)}{\leq} d_b\big(v^{(n)} - v\big) \leq \Lambda\big|v^{(n)} - v\big|_1 \to 0 \\ \text{and} \quad & d_b\big(v\big) - d_b\big(v^{(n)}\big) \overset{(*)}{\leq} d_b\big(v - v^{(n)}\big) \leq \Lambda\big|v - v^{(n)}\big|_1 \to 0, \end{aligned}$$

inequalities (*) coming from

$$d_b(a) - d_b(b) = d_b(a-b+b) - d_b(b) \overset{(6)}{\leq} d_b(a-b) + d_b(b) - d_b(b) = d_b(a-b).$$

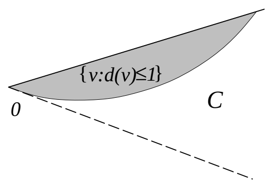
Now consider the unit sphere $\mathcal{S}^D = \big\{u \in \mathbb{R}^D : |u|_2 = 1\big\} \subset \mathbb{R}^D$, well known to be compact. Continuous $d_b\big|_{\mathcal{S}^D}$ therefore attains its minimal value $\tilde{\lambda} := \inf\limits_{u \in \mathcal{S}^D} d_b(u)$ on some $u^{(0)} \in \mathcal{S}^D$. $d_b\big(u^{(0)}\big) = \tilde{\lambda} = 0$ contradicts the nondegeneracy of $d_b$, thus $\tilde{\lambda} > 0$. This means that for arbitrary $v \in \mathbb{R}^D$, $u := v/|v|_2$:

$$d_b(v) = d_b\big(|v|_2 \cdot u\big) = |v|_2 \cdot d_b(u) \geq |v|_2 \cdot \tilde{\lambda} \geq |v|_1 \cdot \tilde{\lambda}\sqrt{D}$$

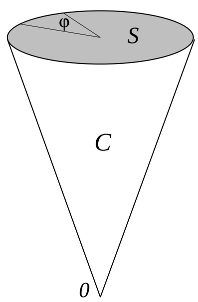
and thus, $\lambda := \sqrt{D} \cdot \tilde{\lambda}$ will do the job.
In the general case, the above considerations show that we find $\lambda_a, \Lambda_a$ and $\lambda_b, \Lambda_b$ to bound $d_a$ and $d_b$ against $|\cdot|_1$ in the sense that $\lambda_a|\cdot|_1 \leq d_a \leq \Lambda_a|\cdot|_1$ and $\lambda_b|\cdot|_1 \leq d_b \leq \Lambda_b|\cdot|_1$. $\Lambda := \Lambda_b/\lambda_a$ and $\lambda := \lambda_b/\Lambda_a$ have the required property:

$$\frac{\lambda_b}{\Lambda_a} d_a \leq \lambda_b|\cdot|_1 \leq d_b \leq \Lambda_b|\cdot|_1 \leq \frac{\Lambda_b}{\lambda_a} d_a \qquad \square$$

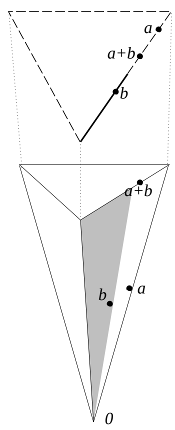
**5.6 Remark:** If $C$ is not closed, $\mathcal{S}^D \cap C$ is not compact. Claim 5.5 therefore does not hold if the distance function is defined only on a convex cone $C \subset \mathbb{R}^2$. The figure to the right depicts the unit sphere of such a $d : C \to \mathbb{R}$ which cannot be bounded from above by $|\cdot|_2$.




But even in case $C$ is closed, there exist counter examples as illustrated to the right: Be $C \subset \mathbb{R}^3$ with circular cross section. For $v \in \mathring{C}$, let $d(v) = |v|_2$. For boundary points $v \in \partial C$, denote $\varphi(v) \in \big[0, 2\pi\big)$ the angle according to the drawing. Then define




$$d(v) = \Lambda\big(\varphi(v)\big) \cdot |v|_2, \qquad \Lambda(\varphi) = \frac{2\pi}{2\pi - \varphi}.$$

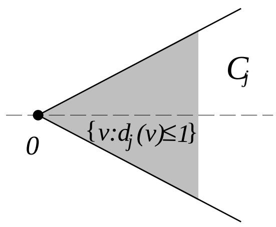



It is important to observe that $d$ indeed fulfills triangle inequality (6) on whole $\overline{C}$: This is due to the fact that points on the boundary of sphere $S$ (the cross section of $C$) cannot be represented as sum of two other points in $S$.
As a consequence, convex $d : C \to \mathbb{R}$ possesses in general no convex extension to the whole space $\mathbb{R}^D$!

Concerning Claim 5.3, the prerequisite $b \in \mathring{C}$ is crucial, too: For $a \in \overline{C}$ and $b \in C$, $a+b$ in general does not lie in $C$ any more!
To this end, consider $C \subset \mathbb{R}^3$ with triangular cross section as sketched to the right. Points $a$ and $b$ are on the same face of $C$, but $a$ lies in the open part of it. And so does $a+b$. $\square$

# 6 Constructing PNGs

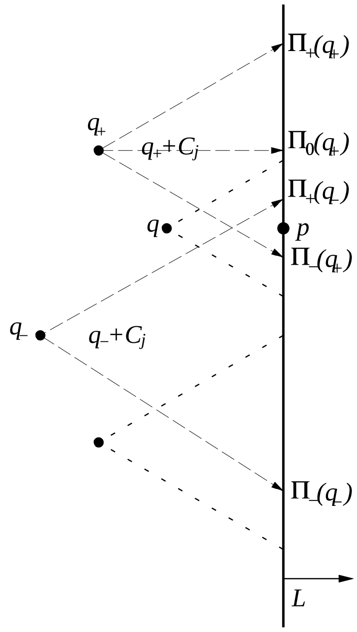
**Proof of Theorem 3.5, first part:** Fix $j$. We will describe an algorithm to compute those arcs $(u,v)$ of $G(\mathcal{C},\tilde{\mathcal{D}};P)$ with $v \in P_j(u)$. According to Equation (5), it then suffices to repeat this process for each $j = 0,1,\ldots,k-1$.




For notational convenience, be the coordinate system such that the symmetry axis of $C_j$ coincides with the $x$-axis. Then $d_j(u) = u_x$ for $u \in C_j$, as a look to the unit sphere of $d_j$ depected in Theorem 3.1 reveals. Sort the points of $P$ in ascending order with respect to their $x$ coordinate — time $\mathcal{O}(n \log n)$ — and let the vertical sweep line $L$ proceed from left to right. We maintain a data structure $S$ for storing all those $u \in P$ lying on the left of $L$ which have not yet got a neighbor $v \in u + C_j$.

Whenever $L$ hits a vertex $p \in P$, we will insert $p$ to $S$, query the data structure about all $q \in S$ such that $p \in q + C_j$, create according edges $(q,p)$, and remove $q$ from $S$: $p$ indeed is closest to $q$. For, suppose $d_j(\tilde{p} - q) = \tilde{p}_x - q_x < p_x - q_x = d_j(p-q)$. Then the line which sweeps $P$ in increasing order of $x$ would have hit $\tilde{p}$ before $p$, thereby having provided $q$ with an edge and removed it from $S$, a contradiction.

Take as $S$ some realization of a dynamic sorted array of $m$ elements (e.g., a balanced binary tree) supporting operations LOCATE, INSERT, and DELETE in (amortized) time $\mathcal{O}(\log m)$.
Each of the $n$ points $p \in P$ is inserted exactly once, hence $m \leq n$, adding to a total time for insertions of $\mathcal{O}(n \log n)$. After any of the $n$ insertion, the above algorithm performs a query of $\mathcal{O}(\log m) + \mathcal{O}(\text{\#elements reported})$, summing up to another $\mathcal{O}(n \log n) + \mathcal{O}(n)$. And finally, $q \in P$ gets deleted at most once: $\mathcal{O}(n \log n)$.
Vertices $u$ which still are in $S$ after the sweeping have $P_j(u) = \emptyset$ and remain without outgoing edge.




Let us now explain how to answer the two-dimensional cone stabbing queries $Q(p) = \{q \in S : p \in q + C_j\}$ required above by means of the one-dimensionally ordered data structure $S$. To this end, be the elements of $S$ sorted with respect to their $y$-coordinates, i.e., the projection $\Pi_0(q)$ of $p$ parallel to the $x$ axis onto the sweep line. $\Pi_0$ has the advantage that is does not change while the sweep line moves and thus can be maintained by data structure $S$. The latter two, on the other hand, do change but they permit to solve the query

$$Q(p) = \{q \in S : \Pi_-(q) \leq \Pi_-(p)\} \cap \{q \in S : \Pi_+(q) \geq \Pi_+(p)\}$$

as follows:

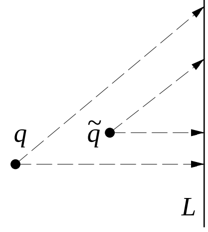
- Find the biggest (w.r.t. $\Pi_-$ ) $q \in S$ which is still smaller than $p$. Call this $q_+$.
- Find the smallest (w.r.t. $\Pi_+$ ) $q \in S$ which is still bigger than $p$. Call this $q_-$.
- Report all vertices $q \in S$ between $q_+$ and $q_-$ (w.r.t. $\Pi_0$ ).

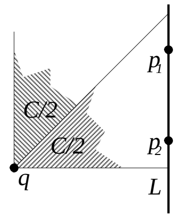
Performing a binary search with respect to one order within items sorted with respect to another usually fails badly. Here, on the contrary, Claim 6.1 guarantees that it does work. The first two steps can therefore be performed in $\mathcal{O}(\log m)$ and the last one indeed returns the elements of $Q(p)$ in output sensitive time. □

**6.1 Claim:** With notions as above, $C \subsetneq \{0\}$, the orders induced by $\Pi_-$ and $\Pi_+$ are weaker than the one induced by $\Pi_0$ in the sense that for $q, \tilde{q} \in S$,

$$\Pi_0(q) \leq \Pi_0(\tilde{q}) \implies \Pi_-(q) \leq \Pi_-(\tilde{q}) \quad \wedge \quad \Pi_+(q) \leq \Pi_+(\tilde{q})$$
$$\Pi_0(q) \geq \Pi_0(\tilde{q}) \implies \Pi_-(q) \geq \Pi_-(\tilde{q}) \quad \wedge \quad \Pi_+(q) \geq \Pi_+(\tilde{q})$$

**Proof:** We consider $\leq$ and suppose $\Pi_0(q) \leq \Pi_0(\tilde{q})$ but $\Pi_+(q) > \Pi_+(\tilde{q})$. From the definition of $\Pi_0$ and $\Pi_+$ as center and upper boundary of $C$, this implies $\tilde{q} \in q + C$. But then, $q \in S$ would have received the neighbor $\tilde{q}$ and been removed from $S$ at that very moment when sweep line $L$ hit $\tilde{q}$ — a contradiction. □





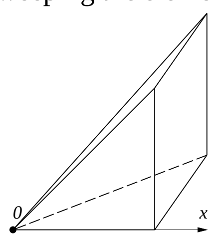

**Proof of Theorem 3.5, second part:** Constructing the PNG of Theorem 3.4 is more difficult for three reasons: Formerly, we could (within $C_j$) identify the line shaped boundary of the distance function's unit sphere with the sweep line and therefore in order of increasing $d_j$ process all vertices in one pass.
This time, *two* lines are needed to cover that boundary. Therefore, divide the quadrant along its diagonal axis: Within each part $C/2$, $d_j$ now has only one segment boundary and can be treated as before. The resulting graph temporarily has outdegree 8, but a subsequent $\mathcal{O}(n)$ processing will compare for each $u$ its two neighbors corresponding to the two parts of $C$ and keep only that arc to the closer one.

The other problem to obey is the boundary of quadrant $C$ and whether it is belongs the the cone or not. This can be taken care of by choosing $q_+$ biggest but smaller *or equal* in the above algorithm and $q_-$ correspondingly.

And third, the tie-breaking-rule (total order) must be applied in case two points are equally close. The latter comes into play when the sweep line simultaneously hits two (or more) vertices $p_1$ and $p_2$: Each $q \in Q(p_1) \cap Q(p_2)$ requires to decide which of $|p_1 - q|_0$ and $|p_2 - q|_0$ is smaller and create either arc $(p_1, q)$ or $(p_2, q)$ accordingly. Luckily, the quadratic time for comparing each $p \in L$ to each $q \in Q(p)$ can be reduced: W.l.o.g. consider the lower $C/2$, the upper one being similar. Now, if the queries $Q(p)$ for different $p \in L$ are processed in increasing order of $p_y$, this will automatically obey the $|\cdot|_0$ condition!
Indeed, the shape of $C/2$ implies that $q_y \leq p_y$ for each $q \in Q(p)$. Furthermore, $|v|_0 = \min\{|v_x|, |v_y|\} = |v_y| = v_y$ for $v \in C/2$. Together, this yields

$$\big|p_2 - q\big|_0 = p_{1,y} - q_y < p_{2,y} - q_y = \big|p_2 - q\big|_0 \qquad \text{for} \quad p_1, p_2 \in L, p_{1,y} < p_{2,y}. \quad \square$$

**Proof of Theorem 3.7:** Like in the two dimensional case, our algorithm will work in phases, one for each cone $C \in \mathcal{C}$ of the covering to compute those arcs $(q, p)$ with $p \in C + q$. Instead of a sweep line $L$, we will employ a plane $H$, sweeping the elements of $P$ in order of increasing $x$-coordinate.
Again, we have to subdivide each cone $C$ in such a way that within each part, the distance function's unit sphere has a planar boundary, i.e., $d\big|_C$ is a projection. To this end, cut octant $C = C_{(+,+,+)}$ into three congruent subcones $C/3 = \{v \in C \,:\, v_y \leq v_x \,\wedge\, v_z \leq v_x\}$ sketched to the right.




And again, too, the rule $|\cdot|_1$ for breaking ties in case $H$ simultaneously hits several vertices $p_1, p_2$ will automatically be fulfilled if these are processed in order of increasing $p_y + p_z$. Put differently, let $H$ sweep $P$ sorted lexicographically with respect to $(x, y+z)$.

It now remains to find a dynamic data structure $S$ for efficiently answering the dimensional cone stabbing queries $Q(p) = \{q \in S : p \in q + C/3\}$. Unfortunately, there is no three dimensional analogon to Claim 6.1: Denote $\Pi_+(q)$ the projection of $q$ parallel to the upper boundary plane of $C/3$ onto sweep plane $H$, i.e. the horizontal line $H \cap (q + \partial^+ C/3)$ and correspondingly $\Pi_-(q)$ for the lower boundary.
Then there exist points $q, \tilde{q}$ such that $\Pi_+(q) < \Pi_+(\tilde{q})$, $\Pi_-(q) > \Pi_-(\tilde{q})$ but neither $q \in \tilde{q} + C$ nor $\tilde{q} \in q + C$: Take the two-dimensional example sketched in Claim 6.1 and

choose the third coordinates of $q$ and $\tilde{q}$ so very different that they do not lie in each other's cone any more!

We will give it another try and analyze the applicability of *range trees* [1]: These dynamic data structures can efficiently answer $D$-dimensional orthogonal range queries parallel to the axises

$$\{q \in S \,:\, a_i \le q_i < b_i,\ i = 1, \ldots, D\} \quad = \quad S \cap \bigtimes_{i=1}^{D} [a_i, b_i) \ =: \ S \cap [a, b)$$

in time $\mathcal{O}\big(\log^D m\big) + \mathcal{O}(\#\text{elements reported})$. Now consider the four faces of $C/3$ and the planes they lie in. Be $u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}$ their normal vectors, oriented in direction of $C/3$, that is

$$\begin{aligned} u^{(1)} &= (0,\ 0,\ 1) && \text{lower boundary plane of } C/3 \\ u^{(2)} &= (1,\ 0, -1)/\sqrt{2} && \text{upper boundary plane of } C/3 \\ u^{(3)} &= (0,\ 1,\ 0) && \text{front boundary plane of } C/3 \\ u^{(4)} &= (1, -1,\ 0)/\sqrt{2} && \text{back boundary plane of } C/3 \end{aligned}$$

Assign to each vertex $p \in P$ the 4-tuple $p^*$ of its distances to these planes

$$p^* = \Big( \sum_i p_i u_i^{(1)}, \sum_i p_i u_i^{(2)}, \sum_i p_i u_i^{(3)}, \sum_i p_i u_i^{(4)} \Big)$$

and observe that $v \in C/3$ if and only if $v^* \in \big[(0,0,0,0), (\infty, \infty, \infty, \infty)\big)$. Thus,

$$q \in Q(p) \quad \Leftrightarrow \quad q \in S \cap (p - C/3) \quad \Leftrightarrow \quad q^* \in S^* \cap \big[-p^*, (-\infty, -\infty, -\infty, -\infty,)\big).$$

So, a four dimensional range tree $S^*$ can be employed to answer the query $Q(p)$. This gives a sweep plane algorithm of time complexity $\mathcal{O}(n \log^4 n)$ — two magnitudes of $\log n$ slower than claimed.

One factor can be removed with the well known *fractional cascading* technique [5, 19]. For the other one, once again subdivide the cone $C/3$ by triangulating its quadratic cross section: The two resulting $C/6$ will have only three boundary planes. Hence, $q^*$ and $S^*$ are three dimensional instead of four. ☐

**6.2 Scholium:**[3] Be $\mathcal{C}$ a partition of $\mathbb{R}^D$ into $k$ convex cones and $\mathcal{D}$ a family of norms $d_j$, $j = 0, \ldots, k-1$. Suppose that each $d_j$ equals the maximum of finitely many projections or, equivalently, its unit sphere is polyhedral.

Then $C_j$ can be subdivided into $\delta_j < \infty$ subcones $C_j/\delta_j$ such that for each one,

- its cross section forms a $(D-1)$ dimensional simplex
- and $d_j\big|_{C_j/\delta_j}$ is a projection.

Furthermore, graph $G(\mathcal{C}, \mathcal{D}; P)$ can be computed from input $P \subset \mathbb{R}^D$ of size $n = \#P$ from performing $\sum_{j=0}^{k-1} \delta_j$ sweep hyperplane passes, each of time $\mathcal{O}\big(n \cdot \log^{D-1} n\big)$. ☐

[3] A scholium is a corollary not to a theorem but to a proof ...

## 7 Proof of Theorem 3.6

We begin with a

**7.1 Remark:** concerning the mapping $\bar{f} : \{+,0,-\}^3 \to \{+,-\}^3$ and its higher dimensional generalizations: This represents a convinient way of specifying for points that are common to the boundary of several octants (in general: hyperquadrants $C_{\bar{\imath}}$, $i \in \{+,-\}^D$) to which one it belongs, thereby turing the covering into a partition. Each possible argument $\bar{k} \in \{+,0,-\}^D$ assigns to a whole face or subface

$$F_{\bar{k}} = \{u \in \mathbb{R}^D : \operatorname{sgn} u = \bar{k}\}, \qquad \operatorname{sgn}(u_1,\ldots,u_d) := (\operatorname{sgn} u_1,\ldots,\operatorname{sgn} u_d),$$

one hyperquadrant $C_{\bar{f}(\bar{k})}$. Denote $\#_0\bar{k} = \operatorname{Card}\{i = 1,\ldots D : k_i = 0\}$, then $F_{\bar{k}}$ has dimension $d - \#_0\bar{k}$.

Alas, not every $\bar{f}$ is admissible for this purpose: The $d$-dimensional (improper) face $F_{\bar{k}} = \overset{\circ}{C}_{\bar{k}}$, $k \in \{+,-\}^D$ must of course be mapped to $C_{\bar{k}}$.

And for example in two dimensions, face $F_{(+,0)}$ — the positive $x$-axis — may not be assigned to the upper left quadrant $C_{(-,+)}$ since it does not belong to its boundary: $\bar{f}(+,0)$ must be either $(+,+)$ or $(+,-)$. This indicates that only zero components of arguments are to be modified. The non-zero ones, $\bar{f}$ must leave unchaned:

$$k_i \neq 0 \qquad \Longrightarrow \qquad f_i(\bar{k}) = k_i \tag{14}$$

As a generalization to this we require that, if a face $F_{\bar{k}}$ is mapped to one hyperquadrant $C_{\bar{i}}$ then all faces $F_{\bar{l}}$ lying w.r.t. inclusion between $F_{\bar{k}}$ and $C_{\bar{i}}$ are so, too:

$$s := f_i(\bar{k}) \qquad \Longrightarrow \qquad \bar{f}(\bar{k}, i = s) = \bar{f}(\bar{k}) \tag{15}$$

with notation $(\bar{k}, i = s) = (k_1,\ldots,k_{i-1},s,k_{i+1},\ldots,k_d)$. Condition (15) for example says that if the positive $x$-axis $F_{(+,0,0)}$ belongs to $C_{(+,+,+)}$ it is not allowed to assign the $xz$-plane $F_{(+,0,+)}$ (the relative topological closure of which $F_{(+,0,0)}$ belongs to) to, lets say, $C_{(+,-,+)}$. In our proof of Theorem 3.6, this kind of sub-/face compatibility condition will ensure the monotony of potential function $\Phi_F$ to hold not only on a $(D-1)$-dimensional face $F$ but also on its boundary, confer Lemma 7.3.

Now each $\bar{f}$ fulfilling the above conditions (14) and (15) induces a permissible partition $\mathcal{C}$ of space into hyperquadrants and vice versa. But $\mathcal{C}$ must also be such that it produces (weak) spanners. A necessary condition to this is, according to Proposition 5.2, that not $\mathcal{C}'$ contains a cycle of length 2. We claim that the latter is equivalent to $\bar{f}$ being antisymmetric:

$$\bar{f}(-\bar{k}) = -\bar{f}(\bar{k}) \qquad \forall \bar{k} \in \{+,0,-\}^D,\ \bar{k} \neq \bar{0}. \tag{16}$$

You will easily verify that the $\bar{f}$ we proposed for $D = 3$ indeed complies with all the above conditions. This can also be seen from its formula representation (18) on page 24. On the other hand, absence of 2-cycles is only necessary: Our proof that $\mathcal{C}$ does yield weak spanners begins with Lemma 7.3.

**7.2 Claim:** The following are equivalent:

a) For each $i = 1, \dots D$, $s \in \{+,-\}$ does $\mathcal{C}'$ as induced by $\mathcal{C}$, $v = (\overline{0}, i = s)$, $S = \{u : u_i = 0\}$ according to Equation (13) in Proposition 5.2, contain no 2-cycle $(a,b,a)$.

b) For each $i = 1, \dots, D$, $s \in \{+,-\}$, $\overline{k} \in \{+,0,-\}^D$,

$$\overline{f}\big(+\overline{k}, i=0\big) \neq \overline{f}\big(+\overline{k}, i=s\big) \qquad \vee \qquad \overline{f}\big(-\overline{k}, i=0\big) \neq \overline{f}\big(-\overline{k}, i=s\big)$$

c) $\overline{f}$ is antisymmetric in the sense of (16).

**Proof:**

**"c ⇒ a" :** Take $i$, $s$ and suppose that $(a,b,a)$ is a cycle of $\mathcal{C}'$, that is there exist $\overline{A}, \overline{B} \in \{+,-\}^D$ such that $\quad v \in \overline{C_{\overline{A}}, C_{\overline{B}}}, \quad a, b \in S,$

$$0 \in a + \big(C_{\overline{A}} \cap S\big)^\circ \subset C_{\overline{A}}, \quad b \in a + C_{\overline{A}}, \qquad 0, a \in b + C_{\overline{B}}.$$

The first implies $A_i = s = B_i$. The latter, by definition of $C_{\overline{\iota}}$ in Equation (11), requires $\overline{A} = \overline{f}\big(\operatorname{sgn}(b-a)\big)$ and $\overline{B} = \overline{f}\big(\operatorname{sgn}(b-a)\big)$. Due to prerequisite (16), $\overline{A} = -\overline{B}$ and in particular $s = A_i = -B_i = s$, a contradiction.

**"a ⇒ b" :** Given $i$, $s$, and $\overline{k}$. Without loss of generality, $k_i = 0$. Let $v := (\overline{0}, i = s)$, $S = \{u \in \mathbb{R}^D : k_i = 0 \Rightarrow u_i = 0\}$, $a := (-\overline{k}, i=0) \in S \ni (+\overline{k}, i=0) =: b$, $\overline{A} = \overline{f}\big(+\overline{k}, i=s\big)$, $\overline{B} = \overline{f}\big(-\overline{k}, i=s\big)$. Note that $v \in \overline{C_{\overline{A}}, C_{\overline{B}}}$ as $A_i = s = B_i$. Suppose b) does not hold. Then

$$\overline{A} = \overline{f}\big(\overline{k}, i=s\big) \overset{(*)}{=} \overline{f}\big(\overline{k}, i=0\big) = \overline{f}\big(\operatorname{sgn}(-a)\big) = \overline{f}\big(\operatorname{sgn}\underbrace{(b-a)}_{=2\overline{k}}\big)$$

and hence $b-a, -a \in C_{\overline{A}}$. Since $S$ is of dimension $d - \#_0\overline{k}$, it even even follows that $-a \in (C_{\overline{A}} \cap S)^\circ$. Similarly, $a - b \in C_{\overline{B}}$, $-b \in (C_{\overline{B}} \cap S)^\circ$. So, $(a,b,a)$ forms a 2-cycle of $\mathcal{C}'$ in contradiction to a).

**"b ⇒ c" :** Suppose that component $f_i$ is not antisymmetric. From all $\overline{k}$ with $f_i\big(-\overline{k}\big) = f_i\big(+\overline{k}\big)$ take one of minimal $\#_0$, i.e., the least number of zeros. Since

$$k_i \neq 0 \qquad \Longrightarrow \quad +k_i \overset{(14)}{=} f_i\big(+\overline{k}\big) \overset{(*)}{=} f_i\big(-\overline{k}\big) \overset{(14)}{=} -k_i,$$

necessarily $k_i = 0$. Set $s := f_i\big(+\overline{k}\big))$ and verify

$$f_i\big(+\overline{k}, i=0\big) = f_i\big(+\overline{k}\big) = s \overset{(15)}{=} f_i\big(+\overline{k}, i=s\big)$$
$$\text{and } f_i\big(-\overline{k}, i=0\big) = f_i\big(-\overline{k}\big) \overset{(*)}{=} f_i\big(+\overline{k}\big) = s \overset{(14)}{=} f_i\big(-\overline{k}, i=s\big)$$

This contradicts b) unless there exists $j \neq i$ such that

$$f_j\big(+\overline{k}, i=0\big) \neq f_j\big(+\overline{k}, i=s\big) \quad \vee \quad f_j\big(-\overline{k}, i=0\big) \neq f_j\big(-\overline{k}, i=s\big).$$

Again necessarily $k_j = 0$. This time, set $\tilde{s} := f_j\big(+\overline{k}\big)$.

In case $f_j$ is not antisymmetric for this $\overline{k}$ either, we will find a third component $\tilde{j}$ different from $i$ and $j$ such that $k_{\tilde{j}} = 0$, and so on. This process obviously terminates after at most $D$ steps, simply because then there are not components left: $\overline{k} = 0$ in contrast to the prerequisite of Equation (16). So without loss of generality be $f_j$ antisymmetric:

$$f_j\big(-\overline{k}, i=0, j=0\big) = f_j\big(-\overline{k}\big) = -f_j\big(+\overline{k}\big) = -f_j\big(+\overline{k}, i=0, j=0\big) = -\tilde{s}.$$

$$\implies \quad f_i\big(+\overline{k}, j=\tilde{s}, i=0\big) \overset{(15)}{=} f_i\big(+\overline{k}\big) = s = \\ = s \overset{(*)}{=} f_i\big(-\overline{k}\big) = f_i\big(-\overline{k}, j=0, i=0\big) \overset{(15)}{=} f_i\big(-\overline{k}, j=-\tilde{s}, i=0\big).$$

$f_i$ is therefore not antisymmetric at argument $(\overline{k}, j=\tilde{s})$, neither. But $\#_0(\overline{k}, j=\tilde{s}) = \#_0\overline{k} - 1$ contradicts the minimality of $\overline{k}$. □

**7.3 Lemma:** Given $\mathcal{C}$, $\tilde{\mathcal{D}}$ as in the prerequisites of Theorem 3.6, $s \in P$ and w.l.o.g. $t = 0 \in P$, $|s|_\infty = 1$. The greedy path in $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ from $s$ to $t$ has nonincreasing $|\cdot|_\infty$. And, while staying on one face $F$ of this cube $Q := \{p : |p|_\infty \leq 1\}$, it is even strictly decreasing with respect to some potential function $\Phi_F$. More precisely, be $a \rightsquigarrow b$ one greedy step and $|a|_\infty = 1 = |b|_\infty$. Then

$$\begin{aligned}
a_x = +1 = b_x &\implies (+b_y, +b_z) < (+a_y, +a_z) \geq (0,0) \\
a_y = +1 = b_y &\implies (+b_z, +b_x) < (+a_z, +a_x) \geq (0,0) \\
a_z = +1 = b_z &\implies (+b_x, +b_y) < (+a_x, +a_y) \geq (0,0) \\
a_x = -1 = b_x &\implies (-b_y, -b_z) < (-a_y, -a_z) \geq (0,0) \\
a_y = -1 = b_y &\implies (-b_z, -b_x) < (-a_z, -a_x) \geq (0,0) \\
a_z = -1 = b_z &\implies (-b_x, -b_y) < (-a_x, -a_y) \geq (0,0) \quad \square
\end{aligned}$$

**7.4 Lemma:** The greedy path will at most once change[4] to a different face

$$F_{\bar{\iota}} = \{q \in \mathbb{R}^3 : |q|_\infty = 1, \operatorname{sgn}(q_i) = s\}, \qquad \bar{\iota} = (i,s) \in \{x,y,z\} \times \{+,-\}$$

of $Q$. More formally, suppose $a \rightsquigarrow b \rightsquigarrow c \rightsquigarrow d$ are subsequent steps of this path with $a, b, c \in \partial Q$, $a \notin F_{\bar{\iota}}$, $b \in F_{\bar{\iota}}$. Then $|d|_\infty < 1$.

**Proof of Theorem 3.6:** Denote $\oplus$ addition modulo 3. Within each face $F_{(i,s)}$, the potential function

$$\Phi_{(i,s)}(v) = \big(|v|_\infty, s \cdot v_{i\oplus 1}, s \cdot v_{i\oplus 2}\big) \qquad \text{lexicographically}$$

strictly decreases. The only 'escape' — changing to another face — can occur at most once. The greedy path thus finally does reach $t$ and remains in $Q$. This implies a weak stretch factor of 2 w.r.t. $|\cdot|_\infty$, and the Euclidean weak stretch facter is at most

$$f^* \quad = \quad \Big\{|a-b|_2/|a|_2 : |a|_\infty = 1 = |b|_\infty\Big\}. \tag{17}$$

[4]Mind that $F_{\bar{\iota}}$ is relatively closed. Therefore, changing can mean "entering new, then leaving old one" in two steps, or "already lying in two faces; leave one, then enter another", or in one step "leave old and enter new".

Equivalence of norms (c.f. Claim 5.5) $|a|_\infty \le |a|_2 \le \sqrt{D}|a|_\infty$ implies $f^* \le 2\sqrt{3}$, but this bound is not tight. For a better one, square both sides of (17) and note that, for symmetry reasons (simultaneously permuting or inverting components of $a$ and $b$), the maximum is w.l.o.g. attained in $a_z = +1$, $0 \le a_x, a_y \le 1$. The extremal location of $b$ is thus $b = (-1,-1,-1)$ and $a_x = a_y$. It therefore remains to maximize the one parameter function

$$[0,1] \ni \lambda \mapsto \left. \frac{|a-b|_2^2}{|a|_2^2} \right|_{\substack{a=(\lambda,\lambda,1) \\ b=(-1,-1,-1)}} = 1 + \frac{4\lambda+5}{2\lambda^2+1}$$

via highschool calculus, obtaining $\lambda_0 = (-5+\sqrt{33})/4$ and $f^* = \sqrt{(7+\sqrt{33})/2} \approx 2.524$. □

**7.5 Scholium:** Suppose that $\overline{f} : \{+,0,-\}^D \to \{+,-\}^D$ is admissible in the sense of Equations (14), (15),(16) and $\tilde{\mathcal{D}}$ a family of $2^D$ total orders extending the norm $|\cdot|_\infty$ such that in $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$, greedy paths visit no vertex more than once.
Then this graph has Euclidean weak stretch

$$\begin{aligned} f^* &\le \sqrt{\max_{0\le\lambda\le 1} \left. \frac{|a-b|_2^2}{|a|_2^2} \right|_{\substack{a=(\lambda,\dots,\lambda,1) \\ b=(-1,\dots,-1,-1)}}} \\ &= \sqrt{1 + \frac{2(d-1)\lambda+d+2}{1+(d-1)\lambda^2}}\Bigg|_{\lambda=\left(\sqrt{d(d+8)}-d-2\right)/2(d-1)} \\ &= \sqrt{\frac{\sqrt{d(d+8)}-4+d}{\sqrt{d(d+8)}-2-d}} \qquad \simeq \qquad \sqrt{d} \end{aligned}$$

**7.6 Claim:** Let $a, b \in \mathbb{R}^3 \setminus \{0\}$, $a \ne b$ and $\mathcal{C}$ as in (11). There exists $C \in \mathcal{C}$ with $0 \in a + C$ <u>and</u> $b \in a + C$ iff for each $i = 0,1,2$ one if the following holds:

a) $a_i \cdot \left(\operatorname{sgn} a_{i\oplus1}, \operatorname{sgn} a_{i\oplus2}\right) \ge (0,0) \quad \wedge \quad (b_i - a_i) \cdot \left(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus1}, \operatorname{sgn} a_{i\oplus2}\right) \le (0,0,0)$
b) $a_i \cdot \left(\operatorname{sgn} a_{i\oplus1}, \operatorname{sgn} a_{i\oplus2}\right) < (0,0) \quad \wedge \quad (b_i - a_i) \cdot \left(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus1}, \operatorname{sgn} a_{i\oplus2}\right) < (0,0,0)$

where inequalities are to be understood with respect to lexicographical order and multiplication performed componentwise.

**7.7 Claim:** Let $a \rightsquigarrow b$ be a greedy step, $|a|_\infty = 1$.

a) $|a-b|_\infty \le 1$. $\quad |a-b|_\infty = 1$, then $|a-b|_1 \le |a|_1$.

b) $a_z \ne 1$, $\; b_z = 1$. $\quad$ Then $a_z = 0$, $\quad |a_x - b_x| + |a_y - b_y| \le |a_x| + |a_y| - 1$.

c) $a_z = 1$, $\; -1 < a_x < 0$, $\; 0 \ne a_y \ne +1$. $\quad$ Then $|b|_\infty < 1$.

d) $a_z = 1$, $\; -1 \le a_x < 0$, $\; 0 < a_y < 1$. $\quad$ Then $|b|_\infty < 1$.

The same holds for coordinates $(x,y,z)$ exchanged with $(y,z,x)$, $(z,x,y)$, $(-x,-y,-z)$, $(-y,-z,-x)$, $(-z,-x,-y)$.

**Proof of Lemma 7.4:** Since Claim 7.6 and Claim 7.7 are invariant under cyclic permutation and inversion of coordinates, we may assume without loss of generality that $\bar{\iota} = (z,+)$, $b_z = 1$. According to Claim 7.7b), $a_z = 0$. Consider the 25 cases $a_x, a_y \in \{-1\}, (-1,0), \{0\}, (0,+1), \{+1\}$:

a) $-1 < a_x < 0, \quad 0 < a_y < 1$ contradicts $a \in \partial Q$.

b) $-1 < a_x < 0,\ a_y = 0$; $-1 < a_x < 0,\ -1 < a_y < 0$; $a_x = 0,\ 0 < a_y < 1$; $a_x = 0,\ a_y = 0$; $a_x = 0,\ -1 < a_y < 0$; $0 < a_x < 1,\ 0 < a_y < 1$; $0 < a_x < 1$, $a_y = 0$; $0 < a_x < 1,\ -1 < a_y < 0$ similarly.

c) $a_x = 1$, $a_y = -1$:
$a \rightsquigarrow b$ greedy, so by definition $\exists C \in \mathcal{C}$: $b, 0 \in a + C$. Since $a_z \cdot (\operatorname{sgn} a_x, \operatorname{sgn} a_y) = (0,0)$, case a) of Claim 7.6 for $i = 2$ implies lexicographically:

$$(0,0,0) \geq (b_z - a_z) \cdot (\operatorname{sgn} a_z, \operatorname{sgn} a_x, \operatorname{sgn} a_y) = (0, b_z - a_z, a_z - b_z)$$

Therefore $b_z \leq a_z$, a contradiction: this case does not occur.

d) $a_x = 1$, $-1 < a_y < 0$; $a_x = 1$, $a_y = 0$; $a_x = 1$, $0 < a_y < 1$; $a_x = 1$, $a_y = 1$; $0 < a_x < 1$, $a_y = 1$; $0 < a_x < 1$, $a_y = -1$; $a_x = 0$, $a_y = 1$ don't either.

e) $a_x = -1$, $a_y = +1$:
This time, case b) of Claim 7.6 for $i = 1$ holds, ensuring $b_y < a_y = 1$. Furthermore, by Claim 7.7a), $1 - b_y \leq |a_y - b_y| \leq |a - b|_\infty \leq 1$. Similar application of Claim 7.6 for $i = 0$ yields $0 \geq b_x > -1$. Thus

$$\big(1 - |b_x|\big) + \big(1 - |b_y|\big) = |a_x - b_x| + |a_y - b_y| \leq |a_x| + |a_y| - 1 = 1,$$

the inequality coming from Claim 7.7b). Hence $|b_x| + |b_y| \geq 1$. As we already know $|b_x|, |b_y| < 1$, this means $b_x \neq 0 \neq b_y$, thereby proving

$$b_z = 1, \qquad -1 < b_x < 0, \qquad 0 < b_y < 1.$$

Put this into Claim 7.7c) to see: $1 > |c|_\infty \geq |d|_\infty$ $\surd$

f) $-1 < a_x < 0$, $a_y = +1$:
Again, Claims 7.6 and 7.7b) say $b_z = 1$, $-1 < b_x < 0$, $0 < b_y < 1$, so $|c|_\infty < 1$.

g) $a_x = -1$, $a_y = 0$:
By $(b_x + 1) + |b_y| \leq 0$ (Claim 7.7b), necessarily $b_x = -1$, $b_y = 0$. Which in turn requires (Claim 7.6) $c_z < 1$, $c_x > -1$, $c_y < 1$. So $|c|_\infty < 1$ unless $c_y = -1$. Analogously to case e), $c_y = -1$ means $-1 < c_x < 0$, $0 < c_z < 1$ and therefore $|d|_\infty < 1$ due to Claim 7.7c) for coordinates $(x,y,z)$ exchanged with $(-y,-z,-x)$. $\surd$

h) $a_x = 0$, $a_y = -1$:
Then necessarily $b = (0,-1,+1)$, $c_x = 1$, $-1 < c_y < 0$, $0 < c_z < 1$, $|d|_\infty < 1$.

i) $-1 < a_x < 0$, $a_y = -1$:
Apply Claim 7.7b) to see $-1 < b_x \le 0$, $-1 \le b_y < 0$. If $b_x$ was $\neq 0$, then Claim 7.7c) would mean $|c|_\infty < 1$. Thus, $b_x = 0$ and (Claim 7.7b) $b_y = -1$. As above, $c_x = 1$, $-1 < c_y < 0$, $0 < c_z < 1$, and $|d|_\infty < 1$.

j) $a_x = -1$, $-1 < a_y < 0$ similarly:
$-1 \le b_x < 0$, $-1 < b_y \le 0$. For $b_y \neq 0$, we have $|c|_\infty < 1$. And for $b_y = 0$, we have $b_x = -1$, implying (like in g) $c_y = -1$, $-1 < c_x < 0$, $0 < c_z < 1$ and $|d|_\infty < 1$.

k) $a_x = -1$, $0 < a_y < 1$:
Claim 7.6 prohibits $b_x = -1$. Claim 7.7b) then infers $-1 < b_x < 0$, $0 < b_y < 1$. And $|c|_\infty < 1$ by Claim 7.7c).

l) $a_x = -1$, $a_y = -1$:
Then $-1 \le b_x \le 0$, $-1 \le b_y \le 0$. Consider sub-cases

i) $b_y \neq 0$, $0 \neq b_x \neq -1 \quad \Rightarrow \quad |c|_\infty < 1$ by applying Claim 7.7d).

ii) $b_y \neq 0$, $b_x = -1 \quad \Rightarrow \quad -1 < c_y \le 0$, $0 \le c_z < 1$ using Claim 7.6 and Claim 7.7a). $|c|_\infty = 1$ requires $c_x = -1$. But then, neither do $b$ nor $c$ leave the face $F_{(x,-)}$ which $a$ started in, preserving strict decrease of the same $\Phi_{(+,-)}$ all the time due to Lemma 7.3.

iii) $b_x = 0 \quad \overset{7.7b)}{\Longrightarrow} \quad b_y = -1$. Refer to case h) to see: $|d|_\infty < 1$.

iv) $b_y = 0 \quad \overset{7.7b)}{\Longrightarrow} \quad b_x = -1$ which transforms to case h) under change of coordinates $(x,y,z) \mapsto (-z,-x,-y)$, and therefore $c_y = -1$, $0 < c_z < 1$, $-1 < c_x < 0$, $|d|_\infty < 1$ as well. □

**Proof of Lemma 7.3:** Consider $a_x = 1 = b_x$, the other cases being similar. Since $(b_x - a_x) \cdot (\cdots) = (0,0,0)$, we have case a) rather than b) of Claim 7.6. Therefore, $a_x \cdot (\operatorname{sgn} a_y, \operatorname{sgn} a_z) \ge (0,0)$ which means $(a_y, a_z) \ge (0,0)$. Now, apply Claim 7.6 to $i = y$ and get

$$(b_y - a_y) \cdot \big(\underbrace{\operatorname{sgn} a_y, \operatorname{sgn} a_z}_{\ge (0,0)}, \underbrace{\operatorname{sgn} a_x}_{=1}\big) \le (0,0,0),$$

hence $b_y \le a_y$. If $b_y < a_y$, we are done.
So, be $b_y = a_y$. This requires $a_y \cdot (\operatorname{sgn} a_z, \operatorname{sgn} a_x) \ge (0,0)$. Combining that with $(a_y, a_z) \ge (0,0)$ implies $a_z \ge 0$ for $a_y \neq 0$ and $a_y = 0$. One more time, look at Claim 7.6 to see $\quad (b_z - a_z) \cdot \big(\underbrace{\operatorname{sgn} a_z}_{\ge 0}, \underbrace{\operatorname{sgn} a_x}_{=1}, \operatorname{sgn} a_y\big) \le (0,0,0) \quad$ and $b_z \le a_z$.
Let's finally exclude the possibility of $b_z = a_z$ by remarking that this means $a \rightsquigarrow b = a$, a self loop. Such, on the other hand, cannot occur in PNGs, as each $C \in \mathcal{C}$ has $0 \notin C$ and thus $u \notin P_j(u)$ in Equation (7). □

**Proof of Claim 7.7:** Since, as a prerequisite to Theorem 3.6, $\tilde{d}_\iota \in \tilde{\mathcal{D}}$ is an extension of $(|\cdot|_\infty, |\cdot|_1)$ w.r.t. lexicographical order, *every* arc — greedy or not — in $G(\mathcal{C}, \tilde{\mathcal{D}}; P)$ trivially obeys a).

For b), $|a|_\infty = 1$ and $a_z \neq 1$ and require $-1 \leq a_z < 1$. $1 - a_z = |b_z - a_z| \leq |b-a|_\infty \overset{a)}{\leq} 1$ further restrict to $0 \leq a_z < 1$. Suppose $a_z > 0$, then application of Claim 7.6 to $i = 3$ yields

$$(1 - a_z, any, any) = (b_z - a_z) \cdot \left( \operatorname{sgn} a_z, \operatorname{sgn} a_x, \operatorname{sgn} a_y \right) \leq (0,0,0)$$

independent of which of the two cases actually holds. Hence, $a_z \geq 1$: a contradiction. The rest of b) is obtained by inserting $a_z = 0$, $b_z = 1$ into a).

$-1 < a_x < 0$ implies $-1 \overset{7.6}{<} b_x \overset{a)}{<} 1$, therefore $|b_x| < 1$. Similarly, cases $-1 < a_y < 0$ and $0 < a_y < 1$ yield $|b_y| < 1$. Analogous arguments in case $a_y = -1$ only gives $-1 \leq b_y \leq 0$, but $b_y = -1$ is ruled out by part b) of Claim 7.6. Finally, from $a_z = 1$ follows $0 \leq b_z \leq 1$, and again $b_z = 1$ prohibited: $|b_z| < 1$, too. Together $|b|_\infty < 1$, the claim of c).

Finally, part d): $0 < a_y < 1$, so $-1 < b_y < 1$. $-1 < a_x < 0$, so $-1 \leq b_x < -1$. Even for $a_x = -1$, $b_x = -1$ is impossible due to Claim 7.6b). The same holds for $b_z = 1$, so $0 \leq b_z < 1$ and $|b|_\infty < 1$. □

**Proof of Claim 7.6:** The reader will easily verify that $\bar{f} : \{+,0,-\}^3 \to \{+,-\}^3$,

$$f_i : \bar{k} = (k_0, k_1, k_3) \mapsto k_{i \oplus j(\bar{k})}, \quad i \oplus j(\bar{k}) := i + \min\left\{ j = 0,1,2 : k_{i \oplus j} \neq 0 \right\} \tag{18}$$

is exactly the one used in Equation (11) and has the following property:

$$f_i(\bar{k}) = s \in \{+,-\} \quad \Leftrightarrow \quad s \cdot (k_i, k_{i\oplus 1}, k_{i \oplus 2}) \geq \bar{0} \quad \Rightarrow \quad s \cdot (k_i, k_{i \oplus 1}) \geq \bar{0}, \tag{19}$$

Remember, that w.r.t. lexicographical order and for $x, y \in \mathbb{R}$, $\bar{u} \in \mathbb{R}^n$, $\bar{v} \in \mathbb{R}^m$:

$$(x,y) \gtreqless (0,0) \quad \Leftrightarrow \quad (\operatorname{sgn} x, y) \gtreqless (0,0) \quad \Leftrightarrow \quad (\operatorname{sgn} x, \operatorname{sgn} y) \gtreqless (0,0) \tag{20}$$

$$\bar{u} \geq \bar{0}, \qquad \bar{v} \geq \bar{0} \qquad \Longrightarrow \qquad \bar{u} \otimes \bar{v} \geq \bar{0} \tag{21}$$

$$\bar{u} > \bar{0}, \qquad \bar{u} \otimes \bar{v} \geq \bar{0} \ \vee \ \bar{v} \otimes \bar{u} \geq \bar{0} \qquad \Rightarrow \qquad \bar{v} \geq \bar{0} \tag{22}$$

$\bar{u} \otimes \bar{v} := \left( u_0 v_0, u_0 v_1, \ldots, u_0 v_m,\ u_1 v_0, u_1 v_1, \ldots, u_1 v_m,\ \ldots\ldots u_n v_0, u_n v_1, \ldots, u_n v_m \right)$.
Thus, for $\bar{s} \in \{+,-\}^3$,

$$\begin{aligned} 0, b \in a + C_{\bar{s}} \ &\overset{(11)}{\iff} \ \forall i = 0,1,2: \ f_i\big(\operatorname{sgn}(b-a)\big) = s_i = f_i\big(\operatorname{sgn}(-a)\big) \\ &\underset{(20)}{\overset{(19)}{\iff}} \ \big(b_i - a_i, b_{i\oplus 1} - a_{i \oplus 1}, b_{i \oplus 2} - a_{i \oplus 2}\big) \cdot s_i =: \vec{u}^i \geq (0,0,0) \\ &\quad \wedge \quad -\big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i \oplus 2}\big) \cdot s_i =: \vec{v}^i \geq (0,0,0) \end{aligned} \tag{23}$$

which by (21) yields, considering only the first 3 components of $\vec{u}^i \otimes \vec{v}^i$:

$$-s_i^2 (b_i - a_i) \cdot \big( \operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i \oplus 2} \big) \geq (0,0,0) \tag{24}$$

and in particular ($s_i^2 = 1$) part a) of the claim. For b), be $a_i \cdot \big( \operatorname{sgn} a_{i \oplus 1}, \operatorname{sgn} a_{i \oplus 2} \big) < (0,0)$. Hence, $0 \neq -\operatorname{sgn} a_i = f_i(-\operatorname{sgn} a) = s_i = -s_{i \oplus 1}$ by definition of $f_i$ and $f_{i \oplus 1}$. It suffices to show $b_i \neq a_i$ since the rest follows from (24).

To this end, suppose on contrary $a_i = b_i$. First component of Equation (23) vanishes:

$$(0,0) \leq s_i \cdot \big( b_{i \oplus 1} - a_{i \oplus 1}, b_{i \oplus 2} - a_{i \oplus 2} \big) = -s_{i \oplus 1} \cdot \big( b_{i \oplus 1} - a_{i \oplus 1}, b_{i \oplus 2} - a_{i \oplus 2} \big)$$

Application of (23) to $i \oplus 1$ instead of $i$ requires the reversed inequality to hold, too:

$$\implies \quad s_{i\oplus 1} \cdot \big(b_{i\oplus 1} - a_{i\oplus 1}, b_{i\oplus 2} - a_{i\oplus 2}\big) = (0,0), \qquad b_i = a_i \text{ by assumption}$$

implying ($s_{i\oplus 1} \neq 0$) $a = b$ in contradiction to the prerequisites.

Be now valid, for each $i = 0,1,2$, one of cases a) and b); $s_i := f_i\big(\operatorname{sgn}(-a)\big)$ and we will prove $-a, b-a \in C_{\bar{s}}^{-}$ by verifying Equation (23). Suppose

$$(b_i - a_i) \cdot \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) < (0,0,0).$$

Multiply by $s_i^2 = 1 \geq 0$ according to (21) to find out

$$s_i \cdot (b_i - a_i) \cdot \underbrace{s_i \cdot \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big)}_{=:\bar{u}} < (0,0,0),$$

$\bar{u}$ being $\leq (0,0,0)$ by definition of $s_i$ and (19). As $\bar{u} \neq (0,0,0)$, we have even $\bar{u} < (0,0,0)$ and may conclude $s_i \cdot (b_i - a_i) > 0$, yielding Equation (23).
If $(b_i - a_i) \cdot \big(\operatorname{sgn} a_i, \operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) = (0,0,0)$, necessarily $b_i = a_i$ and (case a)

$$a_i \cdot \big(\operatorname{sgn} a_{i\oplus 1}, a_{i\oplus 2}\big) \geq (0,0). \tag{25}$$

If $0 \neq \operatorname{sgn} a_i = -s_i$, this means $-s_i \cdot \big(\operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) \geq (0,0)$, and if $0 = \operatorname{sgn} a_i$, $k_i = -\operatorname{sgn} a_i = 0$ reduces Equation (19) to $-s_i \cdot \big(\operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) \geq (0,0)$, too. So, take prerequisite w.r.t. $i \oplus 1$

$$(b_{i\oplus 1} - a_{i\oplus 1}) \cdot \big(\operatorname{sgn} a_{i\oplus 1}, \operatorname{sgn} a_{i\oplus 2}\big) \leq (0,0)$$

and multiply with $s_i^2 = 1$ to end up at $s_i \cdot (b_{i\oplus 1} - a_{i\oplus 1}) \geq (0,0)$. For $a_{i\oplus 1} \neq b_{i\oplus 1}$, this proves (23), so suppose equality. Similar to above, this means (case a)

$$a_{\oplus 1} \cdot \big(\operatorname{sgn} a_{i\oplus 2}, \operatorname{sgn} a_i\big) \geq (0,0) \qquad \text{and} \qquad (b_{i\oplus 2} - a_{i\oplus 2}) \cdot \big(\operatorname{sgn} a_{i\oplus 2}, \operatorname{sgn} a_i\big) \leq (0,0),$$

the latter from prerequisite for $i \oplus 2$. Therefore, $s_{i\oplus 1} \cdot (b_{i\oplus 2} - a_{i\oplus 2}) \geq (0,0)$. Equation (25) finally requires $s_i = s_{i\oplus 1}$ for both $a_i = 0$ and $a_i \neq 0$. □

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