Operations on Signed Distance Function Estimates

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Abstract
This paper investigates signed distance function estimates (SDFEs) that are known to be lower bounds of the actual distance by some constant \( q \in (0, 1) \). Hart demonstrated in Hart (1996) that sphere tracing can be applied to such representations to render implicit surfaces efficiently. We define a broad range of functions as SDFEs that have the same convergence guarantees as exact distance functions have. This definition allows us to focus our investigation on the \( q \in (0, 1) \) approximation-to-distance ratio, as this directly affects the convergence speed of sphere tracing.

Hart also proved that the signed distance to the result of union and intersection can be bounded by the minimum and maximum of the respective signed distance bounds of the arguments. Even though signed distance bounds have found widespread use in the industry, from rendering fonts to modeling freeform shapes Aaltonen (2018), their mathematical background has not been thoroughly investigated to this day.

We provide quantitative extensions to Hart’s results by showing how tighter numerical bounds can be obtained for set-theoretic operations. We devise four types of bounds by partitioning \( \mathbb{E}^3 \) into four different subsets using the argument geometries of these operations. Since the lower bound \( q \) directly affects the efficiency of sphere tracing, this gives valuable insight into the performance of this ray tracing technique.

We also introduce the set-contact smoothness function that describes how two arbitrary point sets make contact at different scales of resolution.

Finally, we conclude that sphere tracing a CSG tree constructed from \( q \in (0, 1) \) bounded signed distance function primitives converge if it converges on the precise distance function of the CSG model.

Keywords:
1. Introduction

Signed distance functions have applications ranging from computer-aided geometric design Malladi et al. (1995) to computer graphics. Even though they offer attractive theoretical and practical properties, constructing exact distance functions to free-form surfaces is intractable in closed form, and their pointwise evaluation on-demand may require an infeasible computation load for real-time applications. As a result, one has to use approximate distance functions either by sampling or constructing bounds to the actual distance. Hart has shown in Hart (1996) that lower bounds of signed distance functions can be used to robustly render geometries using sphere tracing.

Still, real-time graphics applications rely on linear surface approximations predominantly. Although hardware accelerated tessellation units allowed a certain level of adaptation of parametric surfaces, there are very few instances
in the game industry Aaltonen (2018) where the direct visualization of surfaces defined by signed distance function estimates proved to be possible.

Our paper focuses on a particular class of implicit representations, signed distance functions, and their lower bounds. We discuss this class of functions and highlight their theoretical aspects that have practical consequences in rendering.

Overview.

In the next section, we introduce the particular implicit surface rendering problem for which the ray marching algorithm is the brute force approach. In Section 4 and 4.1, signed distance functions will be introduced along with a higher performing algorithm called sphere tracing. Note that there is uncertainty about the exact nomenclature in the literature. Our paper uses a unified framework in that regard. Deviations from some of the more common wordings are highlighted in the sections that follow. Our discussion is based on that of Hart.

A wider family of functions will be introduced and investigated in Section 4.3 called signed distance function estimates (SDFE). Such a distance estimate has convergence guarantees, and yet they can be computed quickly for most surfaces. Our main contribution, a unified investigation of how bounds behave when intersecting SDFEs will be presented throughout the Sections 5, 6, and 7. The intersection theorem is divided into four theo-
remains that describe the behavior of the SDFE on different subsets of space. Section 8 summarizes the four theorems into one intersection theorem. The union counterpart is also presented here.

2. Related work

Angles et al. (2017) on sketch-based implicit skinning use distance functions of primitives, but the implicit blending is applied using falloff functions with finite support. Moreover, the rendering of the representation is not discussed, whereas in this research we aim to provide performance bounds in direct visualization using sphere tracing.

Bloomenthal on implicit surfaces Bloomenthal and Wyvill (1997), is an overview of the modelling and visualization of implicit surfaces; however, signed distance functions, and thus sphere tracing is not discussed.

Wyvill et al. (1999) on extending the CSG tree (relevant for us), Plasko et al. (1995) Function representation in geometric modeling: concepts, implementation and applications - might be a good candidate to hunt notation

Reviewer 3:
Ez benne lett volna az irodalomban, ha nem írtott volna ki mindent, amit nem hivatkoztunk: Wright (2015)
Ez viszont új, de ugyanabból az SG session-ből jön: Evans (2015)

Reviewer 4:
Sherstyuk
Hansen et al. (2007)
Gourmel et al. (2010)

Reviewer 5:
Ricci (1973)
Shapiro (2007)
Gomes et al. (2009)
3. Preliminaries

This section introduces the basics of set-theory and presents the offset operation.

3.1. Notations

Let \((\mathbb{R}^3, d)\) denote a metric space such that \(d : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, +\infty]\) is the Euclidean metric. We use the same symbol for distance to sets, and distance of sets. Empty sets are handled by defining \(\inf \emptyset := +\infty\).

\[
d(., \text{empty}) = \inf
\]

Definition 1 (Distance of sets). Let \(A, B \subseteq \mathbb{R}^3, p \in \mathbb{R}^3\).

\[
d(p, A) := \inf_{a \in A} d(p, a), \quad d(A, B) := \inf_{a \in A, b \in B} d(a, b) = \inf_{a \in A} d(a, B) = \inf_{b \in B} d(A, b).
\]

Definition 2 (Neighborhood). Let us denote the \(r \geq 0\) neighborhood of an element \(p \in \mathbb{R}^3\) as

\[
\mathcal{K}_r(p) := \{x \in \mathbb{R}^3 : d(x, p) < r\}.
\]

\(A \subseteq \mathbb{R}^3\) is open if \(\forall a \in A, \exists \epsilon > 0 : \mathcal{K}_\epsilon(a) \subseteq A\). \(B \subseteq \mathbb{R}^3\) is closed if \(\mathbb{R}^3 \setminus B\) is open. A set is bounded if there exists \(\mathcal{K}_r(p)\) that covers it. The diameter of \(C \subseteq \mathbb{R}^3, C \neq \emptyset\) is

\[
\text{diam } C := \sup \{(d(x, y) : x, y \in C)\}.
\]

The diameter is a real number as long as the set is bounded.

Lemma 1 (Existence of extrema). If \(A \subseteq \mathbb{R}^3\) is closed and \(p \in \mathbb{R}^3\), then \(\exists a \in A : d(p, A) = d(p, a)\). Hutchinson (1994)

Furthermore, we denote the interior of the set \(A \subseteq X\) as \(\text{int } A := \{a \in A \mid \exists \epsilon > 0 : \mathcal{K}_\epsilon(a) \subseteq A\}\). The closure of the set \(A \subseteq X\) is denoted as \(\overline{A} := \{a \in X \mid \forall \epsilon > 0 : \mathcal{K}_\epsilon(a) \cap A \neq \emptyset\}\). The boundary set of \(A\) is denoted by \(\partial A := \overline{A} \setminus \text{int } A\). For any set \(A \subseteq X\), it is clear from the definitions that \(\text{int } A\) is open, \(\overline{A}\) and \(\partial A\) are closed sets.
Proof of \( d(p, D) - c \geq d(p, K_r(D)) \)

(b) Proof of \( d(p, D) - c \leq d(p, K_r(D)) \)

Figure 2: A visualization of the proof for the offset theorem.

3.2. Offset Theorem

Definition 3 (Offset). For any \( D \subseteq \mathbb{R}^3 \) the radius \( r \geq 0 \) closed offset set from \( D \) is defined as

\[
K_r(D) := \{ x \in \mathbb{R}^3 : d(x, D) \leq r \}.
\]

Similarly, the interior of \( K_r(D) \) is denoted as \( K_r^i(D) := \text{int} K_r(D) \). This equals \( \{ x \in \mathbb{R}^3 : d(x, D) < r \} \) if \( r > 0 \).

Theorem 1 (Offset-theorem). If \( D \subseteq \mathbb{R}^3 \) is closed and \( r \geq 0 \), then

\[
\forall p \in \mathbb{R}^3 \setminus K_r(D) : d(p, D) - r = d(p, K_r(D))
\]

The proof can be found in TODO.

As a consequence, offsetting a set twice will be the same as offsetting it once by the sum of the radii, thus

\[
K_{r_1}(K_{r_2}(D)) = K_{r_1 + r_2}(D) \quad \text{for } D \subseteq \mathbb{R}^3, 0 < r_1, r_2.
\]

4. Sphere Tracing

From now on, let us consider surfaces defined implicitly by an \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) function, such that the surface is the \( \{ f \equiv 0 \} := \{ x \in \mathbb{R}^3 : f(x) = 0 \} \) level-set.
A ray is a half line originating from a particular point, for example, the camera. Let us represent rays by their origin $p \in \mathbb{R}^3$ and unit length direction vector $v \in \mathbb{R}^3$, $\|v\|_2 = 1$. Then a ray is written as

$$s(t) := s_{p,v}(t) := p + tv \in \mathbb{R}^3 \quad (t \geq 0).$$

Therefore, the ray-surface intersection problem can be expressed as a root finding problem. We need to find the smallest positive root of the composite function.

$$f \circ s : [0, +\infty) \to \mathbb{R}$$

```latex
\text{Definition 4 (DF).} The function $f : \mathbb{R}^3 \to [0, +\infty)$ is a distance function if $0 \in \mathcal{R}_f$, and

$$f(p) = d(p, \{f \equiv 0\}) \quad (\forall p \in \mathbb{R}^3).$$

Thus, at every sample point the function $f$ evaluates to the distance from the surface it represents implicitly.

\textbf{Example.} The open unit sphere has the following distance function: $f_{\text{sphere}}(p) = d(p, \mathcal{K}_1(0)) = \max(\|p\|_2 - 1, 0)$ ($p \in \mathbb{R}^3$), where $\mathcal{K}$ is was defined in Definition 2.

\textbf{Definition 5 (Unbounding sphere).} The unbounding sphere for the distance function $f : \mathbb{R}^3 \to \mathbb{R}$ at point $p \in \mathbb{R}^3$ is the open sphere $\mathcal{K}_f(p)$. There are no surface points closer to $p$ than $f(p)$, so $\mathcal{K}_f(p) \cap \{f \equiv 0\} = \emptyset$. This property demonstrates that the sphere tracing algorithm can be applied to find the first ray-surface intersection. Algorithm 3 iteratively takes distance-sized steps along the ray; thus no ray-surface intersection is skipped while large empty spaces are traversed quickly.

The sphere tracing algorithm is not optimal; however, better algorithms only differ in a constant factor Keinert et al. (2014); Bálint and Valasek
Figure 3: The sphere tracing algorithm takes distance sized steps, thereby it does not overstep a solution, yet it converges quickly. Each step defines an unbounding sphere that is disjoint from the surface.

(2018). In this paper, we focus on operations on surfaces and their effect on convergence speed, rather than the algorithm itself.

In: \( p, v \in \mathbb{R}^3, |v| = 1 \) ray, \( f : \mathbb{R}^3 \to \mathbb{R} \) SDF estimate

Out: \( t \in [0, +\infty) \) distance traveled along the ray

\[
t := 0; \quad i := 0;
\]

for \( i < i_{\text{max}} \) and \( f(p + t \cdot v) \) not too small; \( i := i + 1; \) do

\[
t := t + f(p + t \cdot v)
\]

end

Algorithm 1: Basic sphere tracing adapted from Hart (1996).

4.1. Signed Distance Functions

Definition 6 (SDF). If \( f : \mathbb{R}^3 \to \mathbb{R} \) function is continuous and \( |f| \) is distance function, then \( f \) is a signed distance function.

Signed distance functions can represent an entire volume by classifying the points of \( \mathbb{R}^3 \) belonging to its ‘interior’ (\( \{ f < 0 \} \)), ‘exterior’ (\( \{ f > 0 \} \)), or to the surface (\( \{ f \equiv 0 \} \)). For example, \( \mathbb{R}^3 \ni p \to ||p||_2 - 1 \in [-1, +\infty) \) is a signed distance function of the unit sphere. Note that distance functions are SDFs as well, but without an ‘interior.’ Moreover, the definition automatically implies that \( 0 \in \mathcal{R}_f \).

4.2. Signed Distance Lower Bounds

In this section, we investigate the how a distance bound can be computed from the Lipschitz constant. Dividing the function by one of its Lipschitz constant is quick way to turn any implicit representation into a sphere traceable
function. We define Signed Distance Lower bound to the same function-set as Hart in Hart (1996); however, our definition differs to lay out the concept for the next section. First, we define the upper distance bounds of a function. Before that, we define the Lipschitz constants of a function and show how they relate to SDFs.

**Definition 7** (Lipschitz constant). Let the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be arbitrary, we define the set of Lipschitz constants as

$$\text{Lip} f := \left\{ L > 0 : \forall x, y \in \mathbb{R}^3 : |f(x) - f(y)| \leq L \cdot d(x, y) \right\}.$$ \hspace{1cm} (2)

The function $f$ is Lipschitz continuous if $\text{Lip} f \neq \emptyset$.

**Lemma 2.** Every signed distance function is Lipschitz continuous, and their smallest Lipschitz constant is one. Formally:

$$\forall f : \mathbb{R}^3 \rightarrow \mathbb{R} \ SDF : \inf \text{Lip} f = \min \text{Lip} f = 1.$$ 

**Definition 8** (Upper distance bounds). For any $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ function, we define the upper distance bound set as

$$\text{Ubd} f := \left\{ Q > 0 : \forall x \in \mathbb{R}^3 : |f(x)| \leq Q \cdot d(x, \{ f = 0 \}) \right\}.$$ 

**Definition 9** (Signed Distance Lower Bound). The $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ function is a signed distance lower bound if $1 \in \text{Ubd} 1$.

If the surface is defined by a function $f$ such that $\text{sgn} \circ f : \mathbb{R}^3 \rightarrow \{-1, 0, 1\}$ is continuous at non-root points, it means that the surface $\{ \text{sgn} \circ f \equiv 0 \} = \{ f \equiv 0 \}$ separates the exterior $\{ \text{sgn} \circ f \equiv 1 \}$ from the interior $\{ \text{sgn} \circ f \equiv -1 \}$. The continuity of $\text{sgn} \circ f$ on $\{ f \neq 0 \}$ is needed in the definition above since otherwise one could choose a point inside the object and change the sign of the SDF value, and that would place the point outside of the object without updating its neighbors. Therefore, continuity is not important, but the following weaker property is.

**Corollary 1** (Bolzano-property). For any signed distance function estimate $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, the following holds:

$$\forall x, y \in \mathbb{R}^3 : f(x) \cdot f(y) \leq 0 \implies \exists z \in \overline{xy} : f(z) = 0.$$
This property means, that if we modify the sphere tracing algorithm in a way that overstepping is possible, then the secant method can still be used to refine the solution.

**Lemma 3.** For any \( f : \mathbb{R}^3 \to \mathbb{R} \), \( \text{Lip} f \subseteq \text{Ubd} f \).

Note that the Lipschitz continuity is a much stronger requirement, for example, if \( \text{Lip} f \neq \emptyset \), then \( f \) is differentiable, yet there are non-continuous \( g : \mathbb{R}^3 \to \mathbb{R} \) functions where \( \text{Ubd} g \neq \emptyset \), e.g.: \( g(x, y, z) := \text{floor}(x) \).

As Hart previously noted in Hart (1996), we can generate a signed distance lower bound the following way.

**Corollary 2.** If the function \( f : \mathbb{R}^3 \to \mathbb{R} \) is Lipschitz continuous and \( 0 \in \mathcal{R}_f \), then for all \( L \in \text{Lip} f \) the function \( \frac{f}{L} \) is a signed distance lower bound.

Using our definition, we can extend this to non-continuous implicit representations, but are bounded.

**Corollary 3.** If \( f : \mathbb{R}^3 \to \mathbb{R} \), \( \text{Ubd} f \neq \emptyset \), and \( 0 \in \mathcal{R}_f \), then for all \( Q \in \text{Ubd} f \) the function \( \frac{f}{Q} \) is a signed distance lower bound.

Both of the above divisions will ensure that the sphere tracing algorithms will not overstep any intersections. To take larger steps, and thus speed up convergence, we have to find smaller \( L \in \text{Lip} f \) or \( Q \in \text{Ubd} f \) values, but this approach would not be able to provide us with a convergence guarantee.

**Remark.** Lipschitz constants are typically computed by finding the largest magnitude of the gradient function; however, Lipschitz-continuity on \( \mathbb{R}^3 \) is a firm requirement. In most cases, \( f \) is Lipschitz-continuous on any compact subset of \( \mathbb{R}^3 \), like algebraic surfaces. We can overcome this limitation by finding the radius of the sphere \( r \) for every \( p \) center for which the distance approximation is precisely this \( r \). Thus, for every \( p \) point, we have to solve the fixpoint equation

\[
F(p) := r = \frac{f(p)}{\min \text{Lip} f|_{\overline{K}_r(p)}} \quad (p \in \mathbb{R}^3),
\]

where \( \overline{K}_r(p) \) is the closed sphere centered at \( p \) with radius \( r \). Once the solution \( r \in \mathbb{R} \) to the above equation has been found, we use that to define the function \( F(p) := r \). Therefore, a fixpoint iteration has to be used for every point in space to evaluate the new \( F \) signed distance lower bound. Usually,
the mathematically and computationally challenging part is computing an acceptable approximation of \( \min \text{Lip} f|_{\mathcal{K},(p)} \).

### 4.3. Signed Distance Function Estimate

Distance lower bounds guarantee that the sphere tracing algorithm will not make mistakes, but it does not guarantee that it will converge. For example, the \( p \to 0 \) function is a signed distance lower bound to every object. Even if we require the function to represent the same object, the convergence is not guaranteed as the function values are allowed to be arbitrarily small far away from the surface. For this reason, we define signed distance function estimates that solve these problems, while it is still a large enough set of functions.

**Definition 10** (Lower distance bounds). For any \( f : \mathbb{R}^3 \to \mathbb{R} \) function, we define the lower distance bound set

\[
\text{Lbd} f := \left\{ q > 0 : \forall x \in \mathbb{R}^3 : |f(x)| \geq q \cdot d(x, \{f = 0\}) \right\}.
\]

**Proposition 1** (Properties). We list a few properties of \( \text{Lbd}, \text{Ubd} \) and \( \text{Lip} \):

1. \( \text{Lbd} f \leq \text{Ubd} f \) meaning \( \forall q \in \text{Lbd} f, \forall Q \in \text{Ubd} f : q \leq Q \)
2. \( \text{Lbd} f \cap \text{Ubd} f \) is empty or has a single element
3. If the sets \( \text{Lip} f; \text{Lbd} f; \) and \( \text{Ubd} f \) are not empty, they are intervals of the kind \( \text{Lip} f = [L, \infty) \) or \( \text{Lip} f = (L, \infty) \); \( \text{Lbd} f = (0, q] \) or \( \text{Lbd} f = (0, q) \), \( \text{Ubd} f = [Q, \infty) \) or \( \text{Ubd} f = (Q, \infty) \), respectively.
4. \( \text{Lip} f \subseteq \text{Ubd} f \) as discussed before
5. \( \forall c \in \mathbb{R} : \text{Lip} cf = |c| \text{Lip} f, \text{Lbd} cf = |c| \text{Lbd} f, \text{Ubd} cf = |c| \text{Ubd} f \)
6. \( \forall c \in \mathbb{R} : \text{Lip}(f + c) = \text{Lip} f; \) however, this property does not apply to \( \text{Lbd} \) and \( \text{Ubd} \).
7. \( \bigcap_{c \in \mathbb{R}} \text{Ubd}(f + c) = \text{Lip} f \)
Definition 11 (SDFE). The function \( f : \mathbb{R}^3 \to \mathbb{R} \) is a signed distance function estimate if \( \text{Lbd} f \neq \emptyset \) and \( 1 \in \text{Ubd} f \).

Note that SDFs are SDFEs as well because if \( f \) is an SDF, then \( \text{Ubd} f = [1, \infty) \) and \( \text{Lbd} f = (0, 1) \). For any \( f : \mathbb{R}^3 \to \mathbb{R} \) function, if \( 1 \in \text{Ubd} f \) it means that \( f \) is a lower distance bound since \( |f(x)| \leq d(x \{ f = 0 \}) \).

If \( f \) is an SDFE, then we call any \( q \in \text{Lbd} f \) the bound of the SDFE \( f \).

Since \( 1 \in \text{Ubd} f \), then \( q \in (0, 1) \). The number \( q \) determines the convergence speed of the sphere tracing algorithms since the step size will be at least \( q \) times the size compared to the exact SDF: \( p \to d(p, \{ f = 0 \}) \).

We explain when a function is an SDFE using the following equivalent definitions.

**Proposition 2** (SDFE equivalence). For any \( f : \mathbb{R}^3 \to \mathbb{R} \) function and \( q \in (0, 1) \) the following statements are equivalent:

1. \( f \) is an SDFE with \( q \in \text{Lbd} f \).
2. \( \text{sgn} \circ f \) is continuous on the \( \{ f \neq 0 \} \) set, \( 0 \in \mathcal{R}_f \), and
   \[
   \forall p \in \mathbb{R}^3 : q \cdot d(p, \{ f \equiv 0 \}) \leq |f(p)| \leq d(p, \{ f \equiv 0 \}) \quad (3)
   \]
3. \( \exists \mu : \mathbb{R}^3 \to [1, \frac{1}{q}] \) bounded function such that \( f \cdot \mu \) is an SDF.

**Proof.** 1.\(\Rightarrow\)2. is trivial.

2.\(\Rightarrow\)3. Since \( \{ f \equiv 0 \} \neq \emptyset \), let
   \[
   \mu(p) := \frac{d(p, \{ f \equiv 0 \})}{|f(p)|} \in \left[ 1, \frac{1}{q} \right] \quad (p \in \mathbb{R}^3),
   \]
   where the range of \( \mu \) is obtained by writing the inequalities in (3) into the nominator above. Trivially, \(|f \cdot \mu|\) is a distance function because the right side of the
   \[
f(p) \cdot \mu(p) = f(p) \cdot \frac{d(p, \{ f \equiv 0 \})}{|f(p)|} = \text{sgn}(f(p)) \cdot d(p, \{ f \equiv 0 \})
   \]
equation is continuous on the \( \{ f \neq 0 \} \) set.

3.\(\Rightarrow\)1. Because \( f \cdot \mu \) SDF is continuous, then the function \( \text{sgn} \circ (f \cdot \mu) = (\text{sgn} \circ f) \cdot (\text{sgn} \circ \mu) = \text{sgn} \circ f \) is also continuous on the set \( \{ f \neq 0 \} \). \( q \in \text{Lbd} f \) and \( 1 \in \text{Ubd} f \) holds because \( \forall p \in \mathbb{R}^3 : \)
   \[
   |f(p)| \cdot 1 \leq |f(p) \cdot \mu(p)| = d(p, \{ f \cdot \mu \equiv 0 \}) = d(p, \{ f \equiv 0 \}) \leq |f(p)| \cdot \frac{1}{q}.
   \]

\[\Box\]
For any function, an unbounding sphere has radius $|f(p)|_Q$, where $Q \in \text{Ubd} \ f$. So what is special about the sphere centered at $p \in \mathbb{R}^3$ and with radius $|f(p)|_q$ ($q \in \text{Lbd} \ f$)? This sphere contains at least one surface point.

Hence, when estimating SDF, our priority is to ensure that $\text{Lbd} \ f \neq \emptyset$: this means that the convergence properties will remain as favourable as that of the sphere tracing. The next most important goal of the SDF estimation is to maximize $\text{sup} \ \text{Lbd} \ f$, because we can also define SDFEs equivalently the following way.

Our paper focuses on giving a single $q \in \text{Lbd} \ f$ for the results of set-operations on SDFEs, and thereby proving convergence of sphere tracing and giving a lower bound for convergence speed.

The addition of $\text{Lbd} \ f \neq \emptyset$ condition means there is a bound $0 < q \in \text{Lbd} \ f$ such that $|f(x)| \geq d(x, \{f = 0\})$ for all $x \in \mathbb{R}^3$.

If the definition holds, we say that $f$ is a signed distance function estimate with bound $K$. The value of $q$ at a given point represents the ratio between the actual and the estimate unbounding sphere radii. Note that Definition 11 also ensures that $0 \in \mathbb{R} f$. The imprecision of the SDFE can be observed in Figure 3 as the distance function is imprecise toward the opening in the surface; and therefore, the unbounding spheres are not touching the surface. Thus, $K$ measures the maximum slowdown of sphere tracing on an SDFE compared to an exact SDF. Note that $K = +\infty$ is allowed. The above leads us to the following equivalent definition.

As a consequence of the definition, $f$ is continuous on the $\{f = 0\}$ surface. However, neither $f$ nor $q$ has to be continuous anywhere else, providing a flexible definition of the estimate. For example, this property allows the usage of bounding volumes to speed up the evaluation time of the SDF. In this case, the SDFE is allowed to jump when passing the surface of a bounding volume used in hierarchical acceleration structures.

5. Intersection-theorem: Part I

For all the theorems that follow, let $f$ and $g$ denote signed distance function estimates (SDFE) with bounds $q_f \in \text{Lbd} \ f$ and $q_g \in \text{Lbd} \ g$ respectively. For simplicity, from this section on, the functions $f : \mathbb{R}^3 \to \mathbb{R}$ may either mean the function or the set $\{f \equiv 0\} \subseteq \mathbb{R}^3$. For example, if we write
\(d(p, f)\), it means \(d(p, \{f \equiv 0\})\). Let us also use the notational shorthands \(f_0^- := \{f \leq 0\}\) and \(f^- := \{f < 0\}\). The \(f_0^+\) and \(f^+\) symbols are analogous. Minimum and maximum on functions are to be interpreted element-wise.

Unquestionably, one of the most important theorems in the field comes from Hart in Hart (1996) that states how set-operations can be applied to objects defined by signed distance bounds \(f\) and \(g\). His theorem states that

**Union:** \(\min(f, g)\) is an SDFE of the \(f_0^- \cup g_0^-\) object.

**Intersection:** \(\max(f, g)\) is an SDFE of the \(f_0^- \cap g_0^-\) object.

**Difference:** \(\max(f, -g)\) is an SDFE of the \(f_0^- \setminus g_0^-\) object.

Despite the practical robustness, Hart’s set theorems do not show that the sphere tracing algorithm will continue to work on the resulting estimate, in fact, the SDFE bound of the resulting function may become infinite.

In the following four theorems, we will provide a finite SDFE bound for the intersection set operation. The intersection theorem was subdivided into four theorems because the conditions and the method used changes significantly depending on the subset of space that we investigate. Therefore, we have four theorems for the same intersection operation; however, the bounds are applied to different subsets of space.

The intersection operations on SDFEs is the maximum of two the arguments, and we denote this function as

\[
h := \max(f, g) = x \rightarrow \max(f(x), g(x)) : \mathbb{R}^3 \rightarrow \mathbb{R}.
\]

Since Hart has proved that \(h\) is a signed lower distance bound, we have to show that \(\text{Lbd} \neq \emptyset\).

For the other set operations on SDFE-s, union and difference, the theorems can be reformulated using the de-Morgan identities while considering the complement object defined by the SDFEs \(-f\) or \(-g\).

**5.1. Intersection Theorem in the Interior**

Since we assumed that \(f\) and \(g\) are SDFEs, we can state our theorem as follows. Figure 4 provides a visual aid.
Figure 4: Intersection theorem when $p$ is inside the intersection

**Theorem 2** (Intersection-theorem: Interior). $h = \max(f, g)$ is an SDFE of the $h^-_0 = f^-_0 \cap g^-_0$ set on the set $h^-_0$ with the bound

$$\min(q_f, q_g) \in \text{Lbd } h$$

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$$\min(\text{Lbd } f, \text{Lbd } g) \subseteq \text{Lbd } h$$
The above means that within the intersection, the resulting SDFE \( h \) keeps the precision of the estimate. For the union operation, the exterior subspace of the union benefits from superior performance.

**Proof.** The fact that \( h \) is a signed distance function estimate was already proven by Hart Hart (1996) in 1994.

First, we prove that if \( f(p) \leq 0 \) and \( g(p) \leq 0 \), then
\[
d(p, h) = \min \left( d(p, f), d(p, g) \right).
\]
(4)

Due to the Bolzano property in Corollary 1, \( d(p, f) = d(p, f_0^+ + 0) \) if \( f(p) \leq 0 \).

So therefore
\[
\min \left( d(p, f), d(p, g) \right) = \min \left( d(p, f_0^+), d(p, g_0^+) \right) = d(p, f_0^+ \cup g_0^+),
\]
because of the definition of distance to a set. Using the de Morgan identity, one can reformulate the above for the intersection
\[
d \left( p, f_0^+ \cup g_0^+ \right) = d \left( p, \mathbb{R}^3 \setminus (f^- \cap g^-) \right) = d \left( p, \mathbb{R}^3 \setminus h^- \right) = d(p, h_0^+) = d(p, h).
\]

In the last equation, the Bolzano property was used once again, and the proof of equation (4) is complete.

Second, let us prove that \( \max(K_f, K_g) \) is a bound of the SDFE \( h \). Because \( p \in h_0^- \),
\[
|h(p)| = \left| \max(f(p), g(p)) \right| = \min \left( |f(p)|, |g(p)| \right).
\]
(5)

Finally, using that \( p \in h_0^- \), so (4) holds, so
\[
d(p, h) = \min \left( d(p, f), d(p, g) \right) \leq \min \left( K_f \cdot |f(p)|, K_g \cdot |g(p)| \right)
\]
\[
\leq \max(K_f, K_g) \cdot \min \left( |f(p)|, |g(p)| \right) = \max(K_f, K_g) \cdot |h(p)|.
\]

Therefore, \( d(p, h) \leq \max(K_f, K_g) \cdot |h(p)| \) holds because of (5).

5.2. Intersection Theorem Far from the Surface

In the previous section, we have seen that the intersection SDFE operations remain as precise as the input distance estimates allow. However, the
Theorem 3 (Intersection-Theorem: Faraway). Let us suppose that $f_0^-$ is bounded and $R > \text{diam } f_0^-$. Then $h$ is an SDFE on the set $\mathbb{R}^3 \setminus \mathcal{K}_R(h_0^-)$ with the bound
\[
\frac{K_f \cdot R}{R - \text{diam } f_0^-} < +\infty.
\]

Proof. First, we derive an upper estimate of $d(p, h_0^-)$ with the exact distance to the bounded set $d(p, f_0^-)$. Since $h_0^- \subseteq f_0^-$,
\[
d(p, h_0^-) \leq d(p, f_0^-) + \text{diam } f_0^- \quad (p \in \mathbb{R}^3).
\]

Figure 5: The theorem when $p$ is far away from the intersection precision can be significantly lower on the exterior. The sections that follow will investigate how the precision drops as the two surfaces meet. In this section, we prove that the bound tends toward that of the smaller object. Figure 5 depicts the situation when the neighborhood of the intersection is excluded from the SDFE bound calculation.
For the rest of the proof, let $p \in \mathbb{R}^3 \setminus K_R(h_0^-)$. Because $R \leq d(p, h_0^-)$, and the assumption $R > \text{diam } f_0^-$, Equation (6) implies

$$0 < R - \text{diam } f_0^- \leq d(p, f_0^-). \quad (7)$$

Second, we express a lower estimate of $\max(f(p), g(p))$ using the distance $d(p, f_0^-)$. Estimating $\max(f(p), g(p))$ using that $f$ and $g$ are SDFEs yields

$$\frac{1}{K_f} \cdot d(p, f_0^-) \leq f(p) \leq \max(f(p), g(p)). \quad (8)$$

Finally, we estimate the SDFE bound from the ratio of (6) and (8):

$$\frac{d(p, h_0^-)}{\max(f(p), g(p))} \leq \frac{d(p, f_0^-) + \text{diam } f_0^-}{K_f^{-1} \cdot d(p, f_0^-)} = K_f \cdot \left(1 + \frac{\text{diam } f_0^-}{d(p, f_0^-)}\right) \leq \frac{K_f \cdot R}{R - \text{diam } f_0^-}. \quad (9)$$

For the last estimation equation (7) was used. \hfill \Box

Note that as $R$ approaches infinity, $K^{(R)}_h \to K_f$. This means that in the distance, the SDFE of the bounded object determines the accuracy of the resulting SDFE.

If none of the surfaces $\{f \leq 0\}$ (red) and $\{g \leq 0\}$ (blue) are finite, then a counterexample in Figure 6 displays a scenario where the bound is infinite. A point from their small, bounded intersection can be arbitrarily distant while remaining halfway between the two infinite planes, thus also resulting in a $K_h = +\infty$ bound.

6. Set-Contact Smoothness

This paper presents two more theorems and proofs for the intersection SDFE operation. The remaining two theorems analyze how the resulting bound behaves close to the intersection but not inside it. This scenario is when the SDFE can lose precision significantly due to the alignment of the two argument sets involved. For this reason, this section introduces the concept behind the set-contact smoothness function. Lemma 4 and Lemma 5 will play an important role in estimating the distance bounds in the next section.
First, we introduce a set operator that subtracts a little more from the other set than the standard set-difference operator. Second, we define a function that describes how two sets in \( \mathbb{R}^3 \) intersect each other at different scales. The notations used in this section will be used in the next sections.

### 6.1. Offset Difference-Set

**Definition 12** (Offset difference of sets). The \( \delta \geq 0 \) offset difference of sets \( F \subseteq \mathbb{R}^3 \) and \( G \subseteq \mathbb{R}^3 \) is

\[
F \setminus_{\delta} G := F \setminus \mathcal{K}_\delta(F \cap G).
\]

**Lemma 4.** Let \( F, G \subseteq \mathbb{R}^3, \delta \geq 0, \) and \( p \in \mathbb{R}^3 \setminus \mathcal{K}_\delta(F \cap G), \) then

\[
\min \left( \frac{\delta}{2}, d(p, F \setminus_{\frac{\delta}{2}} G) \right) \leq d(p, F). \tag{9}
\]

**Proof.** Using the distance definition and that \( F = (F \setminus_{\frac{\delta}{2}} G) \cup (F \cap \mathcal{K}_{\frac{\delta}{2}}(F \cap G)), \)

\[
d(p, F) = \min \left( d(p, F \setminus_{\frac{\delta}{2}} G), d(p, F \cap \mathcal{K}_{\frac{\delta}{2}}(F \cap G)) \right) \tag{10}
\]
holds. However, $F \cap K_{\delta/2} \subseteq K_{\delta/2} (F \cap G)$, so

$$d(p, K_{\delta/2} (F \cap G)) \leq d(p, F \cap K_{\delta/2} (F \cap G)) .$$

(11)

Apply (1) and Theorem 1 for the set $K_{\delta/2} (F \cap G)$ with $r = \frac{\delta}{2}$, so

$$d(p, K_{\delta/2} (F \cap G)) - \frac{\delta}{2} = d(p, K_{\delta} (F \cap G))$$

holds because $p \notin S_{\delta}(F \cap G)$. Therefore

$$\frac{\delta}{2} \leq d(p, K_{\delta/2} (F \cap G)) \leq d(p, F \cap K_{\delta/2} (F \cap G))$$

(12)

Substituting (11) and (12) into (10) imply statement (9).

6.2. Set-Contact Smoothness

**Definition 13.** Let $F, G \subseteq \mathbb{R}^3$ be arbitrary sets. We define their contact smoothness modulus as the function

$$\sigma_{F,G}(\delta) := \min(\delta, d(F \setminus \delta/2 G, G \setminus \delta/2 F)) \quad (\delta \geq 0).$$

The $\sigma_{F,G}$ function describes how well do $F$ and $G$ touch smoothly on different scales, one of which is depicted in Figure 7. For example, if $F$ and $G$ are two perpendicular intersecting lines, then $\sigma_{F,G}(\delta) = \sqrt{2}\delta$.

**Proposition 3 (Properties).** Let $F, G \subseteq \mathbb{R}^3$, then the following holds:

1. $\sigma_{F,G}(0) = 0$
2. $\sigma_{F,G}$ is a monotonically increasing function
3. $\sigma_{F,G}(\delta) \leq \delta$
4. If $F$ and $G$ are closed sets, then $\forall \delta > 0 : \sigma_{F,G}(\delta) \neq 0$

**Proof.** Properties 1 and 3 follow from the definition. Property 2 holds since $F \setminus \delta G$ is also monotonic, so $F \setminus \delta_1 G \subseteq F \setminus \delta_2 G$ if $\delta_1 \leq \delta_2$. The distance of sets is also monotonic, that is, the distance of subsets cannot be less. Property 4 is true since if the remaining sets are non-empty, they remain closed and disjoint. The distance of closed and disjoint sets is non-zero. □
Figure 7: Set-contact smoothness is the distance between the remaining sets for every $\delta \geq 0$.

When one of the sets are not connected, the function

$$\sigma^*_{F,G}(\delta) := d(F \setminus_{\frac{\delta}{2}} G, G \setminus_{\frac{\delta}{2}} F) \quad (\delta \geq 0)$$

can jump to a higher value and stay there until the $K_\delta(F \cap G)$ reaches the next component. Therefore, the $\min(\delta, \cdot)$ is used in the equation allows the definition to make sense when $\sigma^*_{F,G}$ is infinite, and it ensures that properties 1 through 3 hold.

Lemma 5. If $F, G \subseteq \mathbb{R}^3$ are closed, $\delta \geq 0$ and $p \in \mathbb{R}^3 \setminus K_\delta(F \cap G)$, then

$$\frac{1}{2} \sigma_{F,G}(\delta) \leq \max \left( d(p, F), d(p, G) \right)$$

Proof. Using Lemma 4 on both $F \setminus_{\frac{\delta}{2}} G$ and $G \setminus_{\frac{\delta}{2}} F$, and taking the maximum of the inequalities results in

$$\min \left( \frac{\delta}{2}, \max \left( d(p, F \setminus_{\frac{\delta}{2}} G), d(p, G \setminus_{\frac{\delta}{2}} F) \right) \right) \leq \max \left( d(p, F), d(p, G) \right)$$

(13)
Assuming they are not empty $F \setminus \frac{\delta}{2} G$ and $G \setminus \frac{\delta}{2} F$ sets are closed, so there exists $x \in F \setminus \frac{\delta}{2} G$ and $y \in G \setminus \frac{\delta}{2} F$ such that

$$d(p, F \setminus \frac{\delta}{2} G) = d(p, x) \quad \text{and} \quad d(p, G \setminus \frac{\delta}{2} F) = d(p, y)$$

from Lemma 1. Using the distance definition and the triangle inequality in $\mathbb{R}^3$, we can estimate the maximum distance

$$2 \max \left( d(p, F \setminus \frac{\delta}{2} G), d(p, G \setminus \frac{\delta}{2} F) \right) = 2 \max \left( d(p, x), d(p, y) \right) \geq$$

$$\geq d(p, x) + d(p, y) \geq d(x, y) \geq d(F \setminus \frac{\delta}{2} G, G \setminus \frac{\delta}{2} F) = \sigma^{\star}_{F,G} (\delta)$$

Combining (13) and the inequality yields the estimate in the theorem.

If one of the $F \setminus \frac{\delta}{2} G$ and $G \setminus \frac{\delta}{2} F$ sets are empty, then we get a lower estimate of $\frac{\delta}{2}$ from (13). So the Lemma holds in this case because of property 3 in Proposition 3. \qed
7. Intersection Theorem: Part II

In this section, we investigate how the intersection affects sphere tracing near the surface. We compute the signed distance bound in relation to the surface smoothness. First, we present our theorem on the difference set, which is followed by the theorem that provides a bound on the subspace of \( \mathbb{R}^3 \) that is exterior to both argument objects.

7.1. Intersection Theorem on the Difference Set

Theorem 3 proves that the SDFE regains precision further away from the intersection. This section focuses on the behavior of the SDFE close to the intersection surface within the difference set \( g_0^- \setminus \delta f_0^- \) shown in Figure 8.

**Theorem 4** (Intersection-theorem: Difference). For every \( 0 < \delta < R \) the function \( h = \max(f, g) \) is an SDFE on the set \( (g_0^- \setminus \delta f_0^-) \cap K_R(h_0^-) \), with the bound:

\[
\frac{K_f \cdot R}{\sigma_{f_0^- g_0^-}(\delta)} < \infty
\]  

(14)

**Proof.** Since we already know that \( h \) is a distance bound of \( h_0^- \) from Hart (1996), we only have to prove that

\[
\sup \left\{ \frac{d(p, h_0^-)}{h(p)} : p \in g_0^- \setminus \delta f_0^- \right\} < +\infty
\]

Let \( p \in g_0^- \setminus \delta f_0^- \), so \( g(p) \leq 0 \leq f(p) \); therefore \( h(p) = \max(f(p), g(p)) = f(p) \), and

\[
\frac{d(p, h_0^-)}{h(p)} = \frac{d(p, h_0^-)}{f(p)} \leq K_f \cdot \frac{d(p, h_0^-)}{d(p, f_0^-)}
\]

From Lemma 4 with \( F := f_0^- \), \( G := g_0^- \), we can approximate the above further by

\[
K_f \cdot \frac{d(p, h_0^-)}{d(p, f_0^-)} \leq \frac{K_f \cdot d(p, h_0^-)}{\min\left(\frac{\delta}{2}, d(p, f_0^- \setminus \frac{\delta}{2} g_0^-)\right)}
\]  

(15)
Since $p \in g_0^\delta \setminus f_0^\delta \subseteq g_0^\delta \setminus \frac{1}{2} f_0^\delta$, then
\[
d(p, f_0^\delta \setminus \frac{1}{2} g_0^\delta) \geq d(g_0^\delta \setminus f_0^\delta, f_0^\delta \setminus \frac{1}{2} g_0^\delta) \geq d(g_0^\delta \setminus \frac{1}{2} f_0^\delta, f_0^\delta \setminus \frac{1}{2} g_0^\delta) = \sigma_{f_0 \cdot g_0}(\delta),
\]
which implies the following using (15) and Definition 13:
\[
\frac{K_f \cdot d(p, h_0^\delta)}{\min\left(\frac{\delta}{2}, d(p, f_0^\delta \setminus \frac{1}{2} g_0^\delta)\right)} \leq \frac{K_f \cdot d(p, h_0^\delta)}{\sigma_{f_0 \cdot g_0}(\delta)}.
\]
Since $d(p, h_0^\delta) \leq R$, the bound in (14) holds. The bound is finite, because of property 4 in Proposition 3.

Let us look at the bound as we approach the surface. Since $R \to \delta$ is assumed, the bound will tend toward $K_f \cdot \frac{\delta}{\sigma(\delta)}$. Therefore, smoother surface connection results in less precise SDFE.

Remark. Proving that the bound is finite in Theorem 4 is straightforward once we assume that $f_0^\delta$ or $g_0^\delta$ is bounded. If $g_0^\delta$ is bounded, then $d(p, h_0^\delta) \leq \text{diam } g_0^\delta$ is already true since $p \in g_0^\delta$.
If \( f_0^- \) is the bounded set, then according to Theorem 3, we can choose an \( R \) and subdivide \( g_0^- \backslash f_0^- \) along \( \mathcal{K}_R(h_0^-) \). We apply the above Theorem 4 on the \( (g_0^- \backslash f_0^-) \cap \mathcal{K}_R(h_0^-) \) set, and Theorem 3 estimates the bound on the set \( (g_0^- \backslash f_0^-) \cap \mathcal{K}_R(h_0^-) \).

\[
\max \left( \frac{R \cdot K_f}{\sigma_{f_0^- g_0^-}(\delta)}, \frac{K_f \cdot R}{R - \text{diam } f_0^-} \right) < +\infty.
\]

For example, \( R := 2 \cdot \text{diam } f_0^- \) yields a finite bound.

### 7.2. Intersection Theorem on the Exterior

Finally, the only portion of space this paper is yet to investigate is close to the surface, but outside of both argument sets. Figure 9 depicts an example of this case.

**Theorem 5** (Intersection-theorem: Exterior). For every \( 0 < \delta < R \) the function \( h := \max(f, g) \) is an SDFE on the exterior set \( ((f_0^+ \cap g_0^+)) \cap (\mathcal{K}_R(h_0^-) \backslash \mathcal{K}_\delta(h_0^-)) \) with the bound:

\[
\frac{2 \cdot \max(K_f, K_g) \cdot R}{\sigma_{f_0^- g_0^-}(\delta)} < \infty \quad (16)
\]

**Proof.** Let us estimate \( \frac{d(p, h_0^-)}{h(p)} \) from above. Using the SDFE bounds, we can estimate

\[
\max(d(p, f_0^-), d(p, g_0^-)) \leq \max(K_f \cdot f(p), K_g \cdot g(p)) \leq \max(K_f, K_g) \cdot \max(f(p), g(p)).
\]

Applying Lemma 5 to \( f_0^- \) and \( g_0^- \), and using the above we have

\[
\sigma_{f_0^- g_0^-}(\delta) \leq 2 \cdot \max(d(p, f_0^-), d(p, g_0^-)) \leq 2 \cdot \max(K_f, K_g) \cdot h(p).
\]

Using the above and that \( d(p, h_0^-) \leq R \) yields the bound

\[
\frac{d(p, h_0^-)}{h(p)} \leq \frac{2 \cdot \max(K_f, K_g) \cdot d(p, h_0^-)}{\sigma_{f_0^- g_0^-}(\delta)}.
\]
If \( \delta = 0 \) and the surfaces join smoothly, then the sequence \( p_n \) converges faster to \( f_0^- \) and \( g_0^- \) than to \( h_0^- = \{(0,0)\} \).

Since \( d(p, h_0^-) \leq R \), the bound in (16) holds. The bound is finite, because of property 4 in Proposition 3.

Generally, the convergence speed of the sphere tracing algorithm depends on the \( \delta \) 'near-cutoff' distance, on the smaller objects diameter \( \text{diam } f_0^- \), and its SDFE bound \( K_f \). The \( \delta \) appears in sphere tracing implementations as an arbitrarily small value used for a distance threshold under which the computation is terminated. This way, sphere tracing stops when the surface is sufficiently approximated.

If this \( \delta \) near-cutoff distance is zero, the theorem does not hold as illustrated by a counterexample in Figure 10. The

\[
p_n := \left( \frac{1}{n}, \frac{1}{n^2} \right) \in \mathbb{R}^2 \quad (n \in \mathbb{N})
\]

sequence is \( O(n^{-2}) \) close to the two surfaces, but only \( O(n^{-1}) \) close to their intersection set \( h_0^- = \{(0,0)\} \) resulting in \( K_h = +\infty \).

**Remark.** Similarly to the remark at the end of Section 7.1, one can produce a finite bound given that one of the input sets, \( f_0^- \) or \( g_0^- \), is bounded on
the unbounded exterior. For example, using $R := 2 \cdot \text{diam } f_0^-$, the bound becomes:

$$\max \left( \frac{4 \cdot \text{diam } f_0^- \cdot \max(K_f, K_g)}{\sigma_{f_0^- \cdot g_0^-}(\delta)}, \ 2 \cdot K_f \right) < +\infty.$$  

8. Results

Therefore, we can summarize the four intersection theorems in with $R := \text{diam } f_0^-$ where $f_0^-$ covers the set in question and $R := 2 \cdot \text{diam } f_0^-$ when it is unbounded.

**Theorem 6** (Intersection-theorem). Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ are SDFEs, suppose $f_0^-$ is bounded, and let $0 < \delta$. Then, $h := \max(f, g)$ is an SDFE of the $h_0^- = f_0^- \cap g_0^-$ intersection

1. On the $h_0^-$ set with the bound $\max(K_f, K_g)$.
2. On the $g_0^- \setminus \delta f_0^-$ set with the bound

   $$2 \cdot K_f \cdot \max \left( \frac{\text{diam } f_0^-}{\sigma_{f_0^- \cdot g_0^-}(\delta)}, 1 \right).$$

3. On the $f_0^- \setminus \delta g_0^-$ set with the bound $\frac{K_g \cdot \text{diam } f_0^-}{\sigma_{f_0^- \cdot g_0^-}(\delta)}$.
4. On the $(f_0^+ \cap g_0^+) \setminus \delta h_0^-$ set with the bound

   $$\max \left( \frac{4 \cdot \text{diam } f_0^- \cdot \max(K_f, K_g)}{\sigma_{f_0^- \cdot g_0^-}(\delta)}, \ 2 \cdot K_f \right).$$

Therefore, $h = \max(f, g)$ is an SDFE of the intersection on the union of these sets above, namely $\mathbb{R}^3 \setminus (\mathcal{K}_\delta(h_0^-) \setminus h^-)$, with the maximum of the bounds listed. Only the thin $\mathcal{K}_\delta(h_0^-) \setminus h^-$ layer surrounding the surface does not have an SDFE bound. Thus, the intersection operation retains the convergence properties of the sphere tracing algorithm.

Moreover, one can also formulate the corresponding theorem for the union operation by substituting $-f$ and $-g$ to the theorems and combining them.

**Theorem 7** (Union-theorem). Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ are SDFEs, suppose $f_0^-$ is bounded, and let $0 < \delta$. Then, $h := \min(f, g)$ is an SDFE of the $h_0^- = f_0^- \cup g_0^-$ union
1. On the $h_0^+$ set with the bound $\max(K_f, K_g)$.
2. On the $g_0^+ \setminus \delta f_0^+$ set with the bound $\frac{K_f \cdot \text{diam} f_0^-}{\sigma_{f_0^+ g_0^+}(\delta)}$.
3. On the $f_0^+ \setminus \delta g_0^+$ set with the bound 
   \[
   2 \cdot K_g \cdot \max\left(\frac{\text{diam} f_0^-}{\sigma_{f_0^+ g_0^+}(\delta)}, 1\right).
   \]
4. On the $(f_0^- \cap g_0^-) \setminus \delta h_0^+$ set with the bound 
   \[
   \frac{2 \cdot \text{diam} f_0^- \cdot \max(K_f, K_g)}{\sigma_{f_0^+ g_0^+}(\delta)}
   \]

The differences in the theorem are because instead of assuming that $f_0^+$ is finite, we assumed that $f_0^-$ is. Thus, the first, most precise case is on an unbounded volume, but the usage of Theorem 3 is unnecessary here. For the same reason, $(f_0^- \cap g_0^-) \setminus \delta h_0^+ \subseteq f_0^-$ is bounded so the last bound is not complicated further.

The $\sigma_{f_0^+ g_0^+}$ measures the smoothness of the surface of the union as opposed to $\sigma_{f_0^- g_0^-}$, which measures the smoothness of the intersection surface. The thin layer $\mathbb{R}^3 \setminus \delta h_0^+ \cup h_0^+$ on which the SDFE approximation can have an infinite bound lies inside the surface.

9. Conclusion

This paper focused on theoretical properties that have a practical effect on rendering. We presented a conceptual overview of signed distance functions and their lower bounds. The efficiency of the SDFE is characterized by the lower bound since it also bounds the maximum slowdown of the sphere tracing algorithm. We have given two equivalent definitions for SDFE that rightly define the implicitly represented surface, yet the distance bound does not even have to be a continuous function.

Throughout Theorems 2, 3, 4, and 5, we have shown that the intersection operation on SDFEs result in a SDFE with finite bound under certain conditions. Most importantly, the bound is determined by the contact of the argument surfaces in proximity to the resulting intersection surface. Another
important factor is the diameter and radius of the input geometries, and the SDFE bound of the function that defines it implicitly.

In Section 8 we summarized our theoretical results in the intersection and union theorems. We have concluded that apart from the contact smoothness of the argument geometries, the most deciding factor on the convergence speed of the sphere tracing algorithm is the subspace of \( \mathbb{R}^3 \) in which most of the ray tracing occurs.

In summary, we have proved that when these widely used distance bound set-operations are applied, the sphere tracing algorithms will continue to converge. This property was observed but was not explained before. Further research could investigate the optimization of a CSG tree since we formulated exact bounds for each operation. Such a CSG optimization would reorder operations and find better bounding volumes to minimize the resulting SDFE bound and computational cost, so the rays are computed most efficiently.


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