Key Lemma

A T-frame is a sequence of T consecutive time steps. The *frame congestion*, C, in a T-frame is the largest number of packets that traverse any edge in the frame. The *relative congestion*, R in a T-frame is the ratio C/T of the congestion in the frame to the size of the frame.

Lemma 1. *For any set of packets whose paths are edge-simple and have congestion c and dilation d, there is a schedule of length $O(c+d)$ in which packets never wait in queues and in which the relative congestion in any frame of size $\log d$ or greater is at most 1.*

Proof

W.l.o.g. assume $c = d$.

Delays are chosen randomly, independently, and uniformly from the range $[1, \alpha d]$, where α is a sufficiently large constant. A packet moves without waiting after the delay.

Associate the following bad event with each edge g : more than T packets use g in some T-frame, for $T \geq \log d$.

The degree of the dependency graph is bounded by d^2.

The expected number of packets that go through an edge e in a frame of size T is bounded by $\frac{1}{\alpha d} dT = T/\alpha$.

Use the Chernoff bound to show that the probability p that more than T packets is $< e^{-\beta T} < 1/d^{\beta}$, for a sufficiently large β. Since we have at most $(1 + \alpha)d$ T-frames, the bad event for an edge occurs with probability at most $1/d^{\beta-2}$.

Since $4pd^2 < 1$ (for a large enough β) by the Lovasz Local Lemma there is some assignment of delays such that the relative congestion in any frame of size $\log d$ or greater is at most 1.