Case 1: $q_{M+1} = 0$

Let p_i be the probability that i tried to transmit at this step.

$$p_i = \begin{cases} 2^{-b_i} & : \quad 1 \leq i \leq M \\ \lambda_i & : \quad M+1 \leq i \leq N \end{cases}$$

Let $T = \Pi_{i=1}^{N}(1 - p_i)$

$\epsilon_i = \frac{p_i}{1-p_i}$

$Q_i^+ = C\lambda_i$.

Since the ith station transmits successfully with probability $T\epsilon_i$: $Q_i^- = CT\epsilon_i$.

The ith station fails to transmits successfully with probability $(1 - \frac{T}{1-p_i})p_i$. For $i \leq M$,

$$B_i^+ = (1 - \frac{T}{1-p_i})p_i(2^{b_i+1} - 2^{b_i}) = 1 - T(1 + \epsilon_i)$$

For $i > M$

$$B_i^+ = (1 - \frac{T}{1-p_i})\lambda_i = \lambda_i - T\epsilon_i$$

For $i \leq M$

$$B_i^- = \frac{T}{1-p_i}p_i(2^{b_i} - 2^0) = T$$

For $i > M$, $B_i^- = 0$.

We want $E[\Delta\Phi(t)] = C\lambda - CT\sum_{i=1}^{N}\epsilon_i + M - MT - T\sum_{i=1}^{N}\epsilon_i + \sum_{i=M+1}^{N}\lambda_i - MT \geq \delta$

Set $\lambda > \frac{1}{2} + \frac{1}{2C} + \frac{\delta}{C}$, then we need to show that

$$N + M \geq T(2N\sum_{i=1}^{N}\epsilon_i + 2M)$$

Since

$$T = \Pi_{i=1}^{N}(1 - p_i) = \frac{1}{\Pi_{i=1}^{N}(1 + \epsilon_i)}$$

we need to show

$$(N + M)\Pi_{i=1}^{N}(1 + \epsilon_i) \geq 2N \sum_{i=1}^{N} \epsilon_i + 2M$$

Case 1: $\sum_{i=1}^{N} \epsilon_i \geq 1$

$(N + M)\Pi_{i=1}^{N}(1 + \epsilon_i) \geq (N + M)\sum_{i=1}^{N} \epsilon_i \geq 2N \sum_{i=1}^{N} \epsilon_i + 2M$

Case 2: $\sum_{i=1}^{N} \epsilon_i < 1$

$(N + M)\Pi_{i=1}^{N}(1 + \epsilon_i) \geq 2N \sum_{i=1}^{N} \epsilon_i + (1 + \sum_{i=1}^{N} \epsilon_i)M + (1 - \sum_{i=1}^{N} \epsilon_i)N \geq 2N \sum_{i=1}^{N} \epsilon_i + 2M$

Technical Lemma

Lemma 1. *If* $0 \leq \epsilon_i \leq 1$ *then*

$$\Pi_{i=1}^{N}(1+\epsilon_i) \geq 1 + \sum_{i=1}^{N} \epsilon_i$$

$$\Pi_{i=1}^{N}(1+\epsilon_i) \geq 2 \sum_{i=1}^{N} \epsilon_i$$

Proof.

$$\Pi_{i=1}^{N}(1+\epsilon_i) - 2\sum_{i=1}^{N} \epsilon_i \geq \Pi_{i=1}^{N}(1-\epsilon_i) \geq 0$$

□

Case 2: $q_{M+1} \neq 0$

Station $M+1$ succeeds with probability

$W = \Pi_{i=1}^{M}(1-2^{-b_i})\Pi_{i=M+2}^{N}(1-\lambda_i)$

As in case 1, $Q_i^+ = (2N-1)\lambda_i$

$$Q_i^- = \begin{cases} (2N-1)W & : \quad i = M+1 \\ 0 & : \quad otherwise \end{cases}$$

$$B_i^+ = \begin{cases} 2^{-b_i}(2^{b_i+1} - 2^{b_i}) = 1 & : \quad 1 \leq i \leq M \\ 1 - W & : \quad i = M+1 \\ \lambda_i & : \quad M+2 \leq i \leq N \end{cases}$$

$B_i^- = 0$, $i = 1, \ldots, N$.

$E[\Delta\Phi(t)] = (2N-1)\lambda - (2N-1)W + M + 1 - W + \sum_{i=M+2}^{N} \lambda_i$

$\geq (2N-1)\lambda + M + 1 + \frac{(N-M-1)\lambda}{N} - 2N(1-\lambda/N)^{N-M-1} = g(M)$

We will show that $g(M) \geq \delta > 0$ for $\lambda \geq 0.6$.

We show that $g(M) > 0$ in the interval $0 \leq M \leq N-1$, and take δ to be the minimum of g in that interval.

$g(0) = 2N\lambda + (1-\lambda/N) - 2N(1-\lambda/N)^{N-1} \geq 2N(\lambda - e^{-\lambda}) > 0$ for $\lambda \geq 0.6$.

$g(N-1) = 2N\lambda + N(1-\lambda/N) - 2N > 0$ for $\lambda > 1/2$.

Show that $g''(M) < 0$ for $0 \leq M \leq N-1$. Then $g(M) > 0$ in the interval.

Theorem 1. *The exponential backoff protocol is unstable for any* $N > 2$ *when the arrival rate at each station is* λ/N*, for* $\lambda \geq \lambda_0 \approx 0.6$*.*

Theorem 2. *The polynomial backoff protocol is stable for any* $N > 2$*, any* $\alpha > 1$*, and any* λ_i *such that* $\sum_{i=1}^{N} \lambda_i < 1$*.*