M/M/m system

Poisson arrival with rate λ.

Exponential service time with expectation $1/\mu$.

m servers; a customer at the head of the queue is sent to any server that is available.

$p_n =$ probability of n customers in the queue in the steady state.

In the steady state:

$\lambda p_{n-1} = n\mu p_n$, $n \leq m$

$\lambda p_{n-1} = m\mu p_n$, $n > m$

$$p_n = \begin{cases} p_0 \frac{(m\rho)^n}{n!} & : \quad n \leq m \\ p_0 \frac{m^m \rho^n}{m!} & : \quad n > m \end{cases}$$

where $\rho = \frac{\lambda}{m\mu} < 1$ (for stability) and p_0 can be calculated from the condition $\sum_{n=0}^{\infty} p_n = 1$.

Erlang Formula

Let P_Q be the probability that an arrival will find all servers busy and will be forced to wait in queue.

$$P_Q = \sum_{n=m}^{\infty} p_n = \sum_{n=m}^{\infty} \frac{p_0 m^m \rho^n}{m!} = \frac{p_0 (m\rho)^m}{m!(1-\rho)}$$

M/M/∞ system

The global balance equation is

$\lambda p_{n-1} = n\mu p_n$, $n = 1, 2, \ldots$

$p_n = p_0(\frac{\lambda}{\mu})^n \frac{1}{n!}$ $n = 1, 2, \ldots$

$$p_0 = \left[1 + \sum_{n=1}^{\infty} (\frac{\lambda}{\mu})^n \frac{1}{n!}\right]^{-1} = e^{-\lambda/\mu}$$

$p_n = (\frac{\lambda}{\mu})^n \frac{e^{-\lambda/\mu}}{n!}$, $n = 0, 1, \ldots$

That is, in steady state the number in the system is Poisson distributed with parameter λ/μ.

The average number is $N = \lambda/\mu$.

M/G/∞ system

Poisson arrivals with rate λ, service times assumed to be independent with distribution G.

$X(t)$ denote the number of customers in the system at time t.

$\Pr(X(t) = j) = \sum_{n=0}^{\infty} \Pr(X(t) = j|N(t) = n)e^{-\lambda t}\frac{\lambda t}{n!}$

The probability that a customer who arrives at time x will still be present at t is $1 - G(t - x)$.

Let p be the probability that an arbitrary customer that arrives during the interval $[0, t]$ is still in the network at time t.

$$p = \int_0^t (1 - G(t - x))\frac{dx}{t} = \int_0^t (1 - G(x))\frac{dx}{t}$$

independently of the others.

$$P(X(t) = j|N(t) = n) = \begin{cases} \binom{n}{j} p^j (1-p)^{n-j} & : \quad j = 0, 1, . \\ 0 & : \quad j > n \end{cases}$$

Thus,

$$\Pr(X(t) = j) = \sum_{n=j}^{\infty} \binom{n}{j} p^j (1-p)^{n-j} e^{-\lambda t} \frac{\lambda t}{n!}$$

$$= e^{-\lambda t} \frac{(\lambda t p)^j}{j!} \sum_{n=j}^{\infty} \frac{(\lambda t (1-p))^{n-j}}{(n-j)!}$$

$$= e^{-\lambda t p} \frac{(\lambda t p)^j}{j!}$$

$X(t)$ is Poisson distributed with mean $\lambda \int_0^t (1 - G(x)) dx$.

Application: P2P networks

A model for Peer-to-Peer (P2P) networks.

The arrival of new nodes – Poisson distribution with rate λ.

The duration of time a node stays connected to the network is independently and exponentially distributed with parameter μ.

Let G_t be the network at time t. Interested in the evolution in time of the stochastic process $\mathcal{G} = (G_t)_{t\geq 0}$.

The evolution of the size of $\mathcal{G}$ depends only on the ratio $\lambda/\mu = N$, the steady state size of the network. W.l.o.g let $\lambda = 1$.

Theorem 1. *1. For any $t = \Omega(N)$, w.h.p. $|V_t| = \Theta(N)$.*

2. If $\frac{t}{N} \to \infty$ then w.h.p. $|V_t| = N \pm o(N)$.

Proof.

$$p = \frac{1}{t}\int_0^t e^{-(t-\tau)/N} d\tau = \frac{1}{t} N(1 - e^{-t/N}).$$

For $t = \Omega(N)$, $E[|V_t|] = \Theta(N)$.

When $t/N \to \infty$, $E[|V_t|] = N - o(N)$.

Using the Chernoff bound, for $t = \Omega(N)$,

$$Pr\left(||V_t| - E[|V_t|]| \leq \sqrt{bN \log N}\right) \geq 1 - 1/N^c$$

for some constants b and $c > 1$. □

M/M/m/m: The m-server Loss system

Identical to the M/M/m system except that if an arrival finds all servers busy, it is lost.

Blocking probability: Steady-state probability that all circuits are busy.

$\lambda p_{n-1} = n\mu p_n$, $n = 1, 2, \ldots, m$

so, $p_n = p_0(\frac{\lambda}{\mu})^n \frac{1}{n!}$ $n = 1, 2, \ldots, m$

and

$$p_0 = \left[\sum_{n=0}^{m} (\frac{\lambda}{\mu})^n \frac{1}{n!} \right]^{-1}$$

Blocking probability is $p_m = \frac{(\lambda/\mu)^m/m!}{\sum_{n=0}^{m}(\lambda/\mu)^n/N!}$

called **Erlang B formula**.

Holds even for M/G/m/m systems.

M/G/1 system

Poisson arrival with rate λ.

Service time has general distribution. Let X_i is the service time of the ith customer. X_i's are identically distributed, mutually independent, and independent of interarrival times.

$E[X] = 1/\mu =$ average service time.

$E[X^2] =$ second moment service time.

P-K formula

The expected waiting time in queue in a stable $M/G/1$ system is

$W = \frac{\lambda E[X^2]}{2(1-\rho)}$

where $\rho = \lambda/\mu$