# M/G/1 system

Poisson arrival with rate $\lambda$.

Customers are served in the order they arrive. Service time has general distribution. Let $X_i$ is the service time of the $i$th customer. $X_i$'s are identically distributed, mutually independent, and independent of interarrival times.

$E[X] = 1/\mu =$ average service time.

$E[X^2] =$ second moment service time.

# P-K formula

The expected waiting time in queue in a stable $M/G/1$ system is

$$W = \frac{\lambda E[X^2]}{2(1-\rho)}$$

where $\rho = \lambda/\mu$.

The expected number of customers in the queue is

$$N_Q = \lambda W = \frac{\lambda^2 E[X^2]}{2(1-\rho)}$$

# Proof of P-K formula

$W_i$ = Waiting time in queue of the $i$th customer.

$R_i$ = Residual service time seen by the $i$th customer.

$X_i$ = service time of the $i$th customer.

$N_i$ = number of customers found waiting in queue by the $i$th customer upon arrival.

$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j$

$E[W_i] = E[R_i] + E\left[\sum_{j=i-N_i}^{i-1} E[X_j|N_i]\right] = E[R_i] + E[X]E[N_i]$

As $i \to \infty$, we obtain

$W = R + \frac{1}{\mu} N_Q$

where $R = \lim_{n\to\infty} E[R_i]$ = mean residual time.

Thus, $W = \frac{R}{1-\rho}$.

$r(\tau)$ = the remaining time for completion of the customer in service at time $\tau$.

The time average of $r(\tau)$ in the interval $[0,t]$ is

$$\frac{1}{t}\int_0^t r(\tau)d(\tau) = \frac{1}{t}\sum_{i=1}^{M(t)}\frac{1}{2}X_i^2$$

where $M(t)$ is the number of service completions within $[0,t]$.

Assuming the limits below exist:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t r(\tau)d(\tau) = \frac{1}{2}\lim_{t\to\infty}\frac{M(t)}{t}\lim_{t\to\infty}\frac{\sum_{i=1}^{M(t)}X_i^2}{M(t)}$$

Assuming equality (with probability 1) between time-averages and ensemble averages:

$$R = \frac{1}{2}\lambda E[X^2]$$

# Communication Networks Modeled as Network of Queues

Difficulties:

- Even if the input queue is an M/M/1 queue, this is not true for any internal node.
- The inter-arrival times in the second queue are highly correlated with the service time of the first queue.
- Service times of the same packet in different queues are not independent.

# Kleinrock independence assumption

Adopt an $M/M/1$ model for each communication link regardless of the interaction of the traffic on this link with traffic on other links.

A reasonable approximation for networks involving Poisson stream arrivals at the entry points, packet lengths that are nearly exponentially distributed, a densely connected network, and moderate-to-heavy traffic loads.

# Example

Two parallel links between $u$ and $v$.

Poisson arrival to $v$; rate $\lambda$

Exponential service time in the links, independent of arrival times; service rate $\mu$.

Alternative 1: Assign incoming packets to random links.

Expected time in the system $= \frac{2}{2\mu - \lambda}$

Alternative 2:

Assign an incoming packet to the shortest link = one queue.

Expected time in the system $= \frac{2}{(2\mu - \lambda)(1 + \lambda/2\mu)}$.

# Jackson's Theorem

Consider a system of $k$ FIFO queues such that:

1. New customers arrive at queue $i$ according to a Poisson process with rate $r_i$ (some, but all $r_i$'s may be 0).

2. Service at queue $i$ is exponential with rate $\mu_i$.

3. When a customer is served at queue $i$ it proceeds with probability $P_{ij}$ to queue $j$, and with probability $1 - \sum_j P_{ij}$ it leaves the system.

4. All choices are independent of previous choices.

# Flow rate

Let $\lambda_i$ be the arrival rate to queue $i$.

$$\lambda_i = r_i + \sum_{j=1}^{K} \lambda_j P_{ji}$$

$i = 1, \ldots, k$.

Assume that the above system has a unique solution.

Let $\rho_i = \lambda_i / \mu_i$.

Let $n_i(t)$ be the number of customers at queue $i$ at time $t$.

Let $n(t) = (n_1(t), \ldots, n_K(t))$ be the state of the system at time $t$.

**Theorem 1.** *Assume that*

- *The system*

$$\lambda_i = r_i + \sum_{j=1}^{K} \lambda_j P_{ji}, \;\; i = 1, \ldots, K$$

  .

  *has a unique solution.*

- $\rho_i < 1$, *for* $i = 1, \ldots, K$.

*Then the system has a steady state* $(n_1, \ldots, n_K)$, *and*

$$\lim_{t \to \infty} P_t(n) = P(n) = P_1(n_1) \times \ldots P_K(n_K)$$

*and*

$$P_j(n_j) = \rho_j^{n_j}(1 - \rho_j) \;\; n_j \geq 0$$

.

The theorem implies that the system operates as a collection of independent M/M/1 queues, though the arrival process to each of the queues is not necessarily Poisson.

# Application

Routing on the butterfly with Poisson arrivals to the inputs, and exponential transition time through the edges.

Assume that the arrival rate to each of the inputs is $\lambda$.

Then the arrival rate to each of the internal nodes is $\lambda$.

Let $\mu > lambda$.

Then the system is stable, expected time in the system is

$$\leq \frac{\log N}{\mu - lambda}$$