Conditional Expectation

Definition 1. *The random variable* $E[X|Y]$ *is defined to be the random variable* $f(Y)$ *such that* $f(y) = E[X|Y = y]$ *which is the conditional expectation of* X *given that* $Y = y$*:*

$$E[X|Y = y] = \frac{\sum_x xp(x, y)}{\sum_x p(x, y)}$$

Example: Consider independent throws of an unbiased 6-sided die. For $1 \leq i \leq 6$, let X_i denote the number of times the value i appears in n throws of the die. Then

$$E[X_1|X_2] = \frac{n - X_2}{5}$$

$$E[X_1|X_2, X_3] = \frac{n - X_2 - X_3}{4}$$

Property of Conditional Expectation:

$$E[E[X|Y]] = E[X]$$

Martingales

Martingales are useful in handling sums of random variables which are not totally independent.

A sequence of random variables $X_0, X_1, \ldots,$ is said to be a *martingale sequence* if for all $i > 0$, $E[X_i | X_0, \ldots, X_{i-1}] = X_{i-1}$.

Let $X_0, X_1, \ldots$ be a martingale sequence such that for each k,

$$|X_k - X_{k-1}| \leq c_k$$

where c_k may depend on k. Then *Azuma's inequality* gives for all $t \geq 0$ and any $\lambda > 0$,

$$\Pr(|X_t - X_0| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{2\sum_{k=1}^{t} c_k^2}} \tag{1}$$

If $|X_k - X_{k-1}| \leq c$ where c is independent of k, then for all $t \geq 0$ and any $\lambda > 0$,

$$\Pr(|X_t - X_0| \geq \lambda c\sqrt{t}) \leq 2e^{-\lambda^2/2}$$

"Method of Bounded differences".

Doob Martingale

Let $Z_1, Z_2, \ldots, Z_n$ be *any* sequence of random variables and let X be any random variable.

Define the random variable $X_i = E[X | Z_1, \ldots, Z_i]$, i.e., the conditional expectation of X conditioned on the variables Z_1 to Z_i.

Then $X_0 = E[X], X_1, \ldots, X_n$ form a martingale sequence (Doob martingale).

Lipschitz condition

Definition 2. *Let* $f : D_1 \times \ldots \times D_n \rightarrow R$ *be a real-valued function with* n *arguments from possibly distinct domains. The function* f *is said to satisfy the Lipschitz condition if for any* $x_1 \in D_1, \ldots, x_n \in D_n$*, any* $i \in \{1, \ldots, n\}$*, and any* $y_i \in D_i$*,* $|f(x_1, \ldots x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \leq 1$

We use an exposure martingale to prove that $\Gamma_i(v)$ is concentrated around its mean with high probability.

Let $w_1, w_2, ...$ be an enumeration of the nodes in W. Define an martingale $Z_0, Z_1, ...$, such that $Z_0 = E[Y]$, $Z_i = E[Y \mid N(w_1), ..., N(w_i)]$, $Z_w = Y$. Since the degree of all nodes is bounded by C, a node w_i can connect to no more than C nodes outside W. Thus, $|Z_i - Z_{i-1}| < C$.

Using Azuma's inequality it follows that that for sufficiently large constant d,

$$Pr\{|Y - E[Y]| \geq \frac{f}{8}\frac{\sqrt{w}}{C} C\sqrt{w}\} \leq 2e^{-\frac{f^2}{128C^2}w} \leq 1/N^5.$$

Our goal is to show that w.h.p the distance between any two c-nodes is $O(\log N)$.

Consider any two c-nodes v and u.

By applying expansion lemma repeatedly $O(\log N)$ times we have with probability $1 - O(\frac{\log N}{N^5})$, for some $k_v, k_u = O(\log N)$, $|\Gamma_{k_v}(v)| \geq \sqrt{N} \log N$ and $|\Gamma_{k_u}(u)| \geq \sqrt{N} \log N$.

The probability that $\Gamma_{k_v}(v)$ and $\Gamma_{k_u}(u)$ are disjoint and not connected by an edge is bounded by $(1 - f/2N)^{N \log^2 N}$.

Thus with probability $1 - O(\frac{\log N}{N^5})$ an arbitrary pair of nodes u and v are connected by a path of length $O(\log N)$ in G_t.

Summing the failure probability over all $\binom{n}{2}$ pairs it follows that w.h.p. any pair of nodes in G_t is connected by a path of length $O(\log N)$.