# Min-Cut Algorithm

**Input:** A $n$ node graph
**Output:** A minimal set of edges that disconnects the graph.

1. **repeat** $n - 2$ **times**:
   (a) Pick an edge uniformly at random.
   (b) Contract the edge.

**endrepeat**

2. **output** the set of edges connecting the two remaining vertices.

**Theorem 1.** *The algorithm outputs a min-cut set of edges with probability* $\geq \frac{2}{n(n-1)}$.

**Lemma 1.** *Contraction operation (step 1(b)) does not reduce the size of the min-cut set.*

**Proof.** Every cut set in the new graph is a cut set in the original graph. $\square$

# Analysis of the Algorithm

Assume that the graph has a min-cut set of $k$ edges.

We compute the probability of finding one such set $C$.

**Lemma 2.** *Let $C$ be a min-cut set of $G$. If the run of the algorithm did not contract any edge of $C$, it also did not eliminate any edge of $C$.*

Since the minimum cut-set has $k$ edges, all vertices have degree $\geq k$, and the graph has $\geq nk/2$ edges.

Let $E_i =$ "the edge contracted in iteration $i$ is not in $C$".

We want

$\Pr(\cap_{i=1}^{n-2} E_i) = \Pr(E_1)\Pr(E_2|E_1)\Pr(E_3|E_2 \cap E_1)\ldots\Pr(E_{n-2}|\cap_{i=1}^{n-3} E_i)$.

$\Pr(E_1) \geq 1 - \frac{k}{nk/2} = 1 - \frac{2}{n}$

$\Pr(E_2|E_1) \geq 1 - \frac{k}{(n-1)k/2} = 1 - \frac{2}{n-1}$

$\Pr(E_i|\cap_{j=1}^{i-1} E_j) \geq 1 - \frac{2}{n-i+1}$

Thus, $\Pr(\cap_{i=1}^{n-2} E_i) \geq \Pi_{i=1}^{n-2}\left(1 - \frac{2}{n-i+1}\right) = \Pi_{i=1}^{n-2}\left(\frac{n-i-1}{n-i+1}\right) = \frac{2}{n(n-1)}$

# Min-Cut: High Probability Algorithm

**Input:** A $n$ node graph
**Output:** A minimal set of edges that disconnects the graph.

1.**for** $i = 1$ **to** $n^2 \ln n$
    1.1. **repeat** $n - 2$ **times**:
        (a) Pick an edge uniformly at random.
        (b) Contract the edge.
    **endrepeat**
    1.2 Let $C_i$ be the set of edges connecting the two remaining vertices.
**endfor**
2. **output** the set with the minimum size among the $C_i$'s.

**Theorem 2.** *The high probability algorithm outputs a min-cut set with probability at least* $1 - 1/n^2$.

# Random Variable

Let $(\mathcal{S}, Pr)$ be a discrete probability space.

Let $V$ be a set of values.

A random variable $X$ defined on $(\mathcal{S}, Pr)$ is a function

$$X : \mathcal{S} \to V$$

Let $\mathcal{E}(r) = \{s \in \mathcal{S} \mid X(s) = r\}$

$$Pr(X = r) \;=\; Pr(\mathcal{E}(r)) = \sum_{s \in \mathcal{E}(r)} Pr(s).$$

Two random variables $X$ and $Y$ (defined on the same sample space) are called independent if for all $x$ and $y$

$$\Pr\{X = x \text{ and } Y = y\} = \Pr\{X = x\} \Pr\{Y = y\}$$

# Expectation

The **expectation** of a discrete random variable $X$:

$$E[X] \;=\; \sum_{i \in range((X))} i \cdot Pr(X = i).$$

# Linearity of Expectation

**Theorem 3.** *For any two random variables $X$ and $Y$*

$$E[X+Y] \;=\; E[X] \;+\; E[Y].$$

**Proof.**

$$E[X+Y] \;=$$

$$\sum_{i\in range(X)}\;\sum_{j\in range(Y)}(i+j)Pr((X=i)\cap(Y=j)) =$$

$$\sum_i\sum_j iPr((X=i)\cap(Y=j)) +$$

$$\sum_j\sum_i jPr((X=i)\cap(Y=j)) =$$

$$\sum_i iPr(X=i) + \sum_j jPr(Y=j).$$

□

(Since we sum over all possible choices of $i$ ($j$).)

**Lemma 3.** *If* $E_1, E_2, ...., E_k$ *are disjoint events such that* $\sum_{i=1}^{k} Pr(E_i) = 1$ *then for any event* $B$*,*

$$\sum_{i=1}^{k} Pr(B \cap E_i) = Pr(B).$$