A Probabilistic Recurrence

Let $g(x)$ be a monotone non-decreasing function from R^+ to R^+. Consider a particle whose position changes at discrete time steps and is always at a positive integer. If the particle is currently at position $m > 1$, it proceeds at the next step to position $m - X$, where X is a random variable over integers $1, \ldots, m - 1$. We are only given that $E[X] \geq g(m)$ and that X is chosen independently of the past.

Assuming the particle starts at position n, what is the expected number of steps before it reaches position 1 ?

Theorem 1. *Let T be the random variable denoting the number of steps in which the particle reaches the position 1. Then $E[T] \leq \int_1^n dx/g(x)$.*

Proof. By induction on n.

Suppose the theorem holds for values of m smaller than n. Let $f(m) = \int_i^m dx/g(x)$ for $m \geq 1$.

Consider the first step, during which the particle proceeds from position n to position $n - X$, where X is chosen from a distribution for which $E[X] \geq g(n)$.

We have

$$
\begin{aligned}
E[T] &\leq 1 + E[f(n-X)] \\
&= 1 + E[\int_1^n dy/g(y) - \int_{n-X}^n dy/g(y)] \\
&= 1 + f(n) - E[\int_{n-X}^n dy/g(y)] \\
&\leq 1 + f(n) - E[\int_{n-X}^n dy/g(n)] \\
&= 1 + f(n) - E[X]/g(n) \leq f(n)
\end{aligned}
$$

□

Bounding Deviation from Expectation

Theorem 2. [Markov's Inequality] *For any non-negative random variable*

$$\Pr(X > a) \leq E[X]/a$$

Example: What is the probability of getting more than $3N/4$ heads in N coin flips?

$\leq \frac{N/2}{3N/4} \leq 2/3$

Variance

The variance of a r.v. X is

$$Var[X] = E[(X - E[X])^2] = E[X^2] - E^2[X]$$

The standard deviation of a r.v. X is

$$\sigma(X) = \sqrt{Var[X]}$$

Chebyshev's Inequality

Theorem 3. *For* any *random variable*

$$\Pr(|X - E[X]| \geq a) \leq \frac{Var[X]}{a^2}$$

$$\Pr(|X - E[X]| \geq a\sigma[X]) \leq 1/a^2$$

$$\Pr(|X - E[X]| \geq \epsilon E[X]) \leq \frac{Var[X]}{\epsilon^2 E^2[X]}$$