Lecture Note 17

Kyung Lyung Hong

hong2@cs.purdue.edu

1 M/G/1 system

- M(memoryless) : Possion arrival process with rate λ.
- G(general) : general service time distribution, mean $X = 1/\mu$.
  Let X_i is the service time of the ith customer. X_i's are identically distributed, mutually independent, and independent of interarrival times.
- 1 : single server, load $\rho = \lambda X$ (in a stable queue one has $\rho < 1$)
  The number of customers in the system, $N(t)$, does not now constitute a Markov process.

$$E[X] = 1/\mu = \textit{average service time.}$$

$$E[X^2] = \textit{second moment service time.}$$

2 P-K formula

The expected waiting time in queue in a stable M/G/1 system is

$$W = \frac{\lambda E[X^2]}{2(1-\rho)}, \quad \textit{where } \rho = \lambda/\mu.$$

The expected number of customers in the queue is

$$N_Q = \lambda W = \frac{\lambda^2 E[X^2]}{2(1-\rho)}$$

3 Proof of P-K formula

- W_i = Waiting time in queue of the ith customer.
- R_i = Residual service time seen by the ith customer.
- X_i = service time of the ith customer.
- N_i = number of customers found waiting in queue by the ith customer upon arrival.

$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j$$

$$E[W_i] = E[R_i] + E\left[\sum_{j=i-N_i}^{i-1} E[X_j|N_i]\right] = E[R_i] + E[X]E[N_i],$$

$$(\ use \ \ E[E[X|Y]] = E[X]\ )$$

As $i \to \infty$, we obtain

$$W = R + \frac{1}{\mu}N_Q = R + \frac{1}{\mu}\lambda W$$

, where $R = \lim_{n\to\infty} E[R_i] = mean\ residual\ time.$

$$Thus,\ W = \frac{R}{1-\rho}.$$

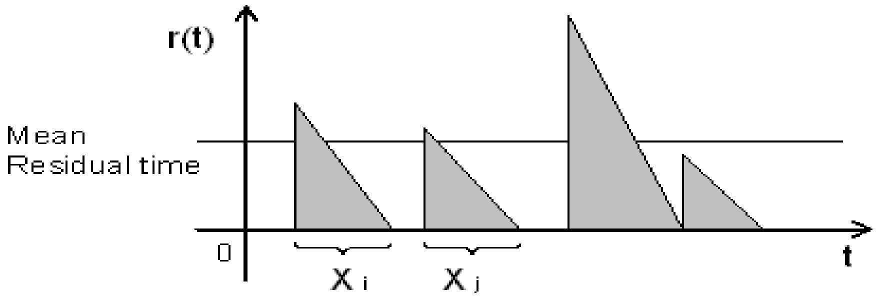


Figure 1: Residual Time.

The Figure 1. represents the evolutionof the unfinished work in the server, r(t), as a function of time.
$r(\tau)$ = the remaining time for completion of the customer in service at time τ.
The time average of $r(\tau)$ in the interval $[0, t]$ can be calculated by dividing the sum of the areas of the triangles by the length of the interval.

$$\frac{1}{t}\int_0^t r(\tau)d(\tau) = \frac{1}{t}\sum_{i=1}^{M(t)} \frac{1}{2}X_i^2$$

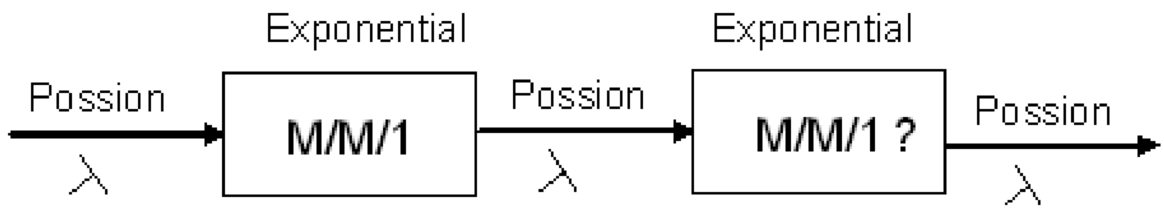
where $M(t)$ is the number of service completions within $[0, t]$.
Assuming the limits below exist:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t r(\tau)d(\tau) = \frac{1}{2}\lim_{t\to\infty}\frac{M(t)}{t}\lim_{t\to\infty}\frac{\sum_{i=1}^{M(t)} X_i^2}{M(t)}$$

$$(\frac{M(t)}{t} = arrival\ rate\ = \ departure\ rate\ = \lambda,$$

$$\frac{\sum_{i=1}^{M(t)} X_i^2}{M(t)} = long\ term\ average\ of\ residual\ time = E[X^2])$$

Assuming equality (with probability 1) between time-averages and ensemble averages:



$$R = \frac{1}{2}\lambda E[X^2]W$$

$$= \frac{\lambda E[X^2]}{2(1-\rho}$$

4 Communication Networks Modeled as Network of Queues

- The output process from an M/M/1 queue is a Possoin process of the same rate λ as the input.
- Even if the input queue is an M/M/1 queue, this is not true for any internal node.
- The inter-arrival times in the second queue are highly correlated with the service time of the first queue.
  ex) if service time at Q_1 increases, no waiting time required in Q_2.
- Service times of the same packet in different queues are not independent.

5 Kleinrock independence assumption

Adopt an M/M/1 model for each communication link regardless of the interaction of the traffic on this link with traffic on other links.

A reasonable approximation for networks involving Poisson stream arrivals at the entry points, packet lengths that are nearly exponentially distributed, a densely connected network, and moderate-to-heavy traffic loads.

6 Example

Two parallel links between u and v.
Poisson arrival to v; rate λ.
Exponential service time in the links, independent of arrival times; service rate μ.

- Alternative 1: Assign incoming packets to random links. (if split possion process randomly, both are poisson process.)
  Expected time in the system = $\frac{2}{2\mu-\lambda}$
- Alternative 2: Assign an incoming packet to the shortest link = one queue.
  Expected time in the system = $\frac{2}{(2\mu-\lambda)(1+\lambda/2\mu)}$.

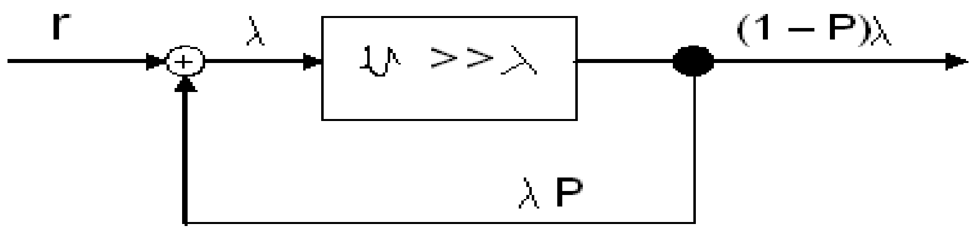
7 Jackson's Theorem

Consider a system of k FIFO queues such that: (k independent M/M/1 queues)

1. New customers arrive at queue i according to a Poisson process with rate r_i (some, but all r_i's may be 0).
2. Service at queue i is exponential with rate μ_i.
3. When a customer is served at queue i it proceeds with probability P_{ij} to queue j, and with probability $1 - \sum_j P_{ij}$ it leaves the system.
4. All choices are independent of previous choices.

8 Flow rate

- cusomers are processed fast ($\mu \gg \lambda$).



- customers exit with probability $(1 - P)$.
  - customers return to queue with probability P
  - $\lambda = r + P\lambda \Rightarrow \lambda = \frac{r}{(1-P)}$
- When P is large, each external arrival is followed by a burst of internal arrivals.
  - Arrivals to queue are not Possion.

Let λ_i be the arrival rate to queue i.

$$\lambda_i = r_i + \sum_{j=1}^{K} \lambda_j P_{ji} i = 1, \ldots, k.$$

Assume that the above system has a unique solution.
($\Rightarrow$ Every packet should leave system eventually though internal arrival is not M/M/1.)
Let $\rho_i = \lambda_i / \mu_i$.
Let $n_i(t)$ be the number of customers at queue i at time t.
Let $n(t) = (n_1(t), \ldots, n_K(t))$ be the state of the system at time t.

8.1 Theorem 1.

Assume that

- The system

$$\lambda_i = r_i + \sum_{j=1}^{K} \lambda_j P_{ji}, \quad i = 1, \ldots, K \quad \textit{has a unique solution.}$$

- $\rho_i < 1$, for $i = 1, \ldots, K$.

Then the system has a steady state $(n_1, \ldots, n_K)$, and

$$\lim_{t\to\infty} P_t(n) = P(n) = P_1(n_1) \times \ldots P_K(n_K) \textit{ and}$$

$$P_j(n_j) = \rho_j^{n_j}(1-\rho_j) \;\; n_j \Rightarrow \lim_{t\to\infty} P_t(n) = \prod_i^k \rho_j^{n_j}(1-\rho_j) \;\; n_j \geq 0.$$

The theorem implies that in steady state the state of node $i(n_i)$ is independent of the states of all other nodes (at a given time)

- The system operates as a collection of independent M/M/1 queues,
- The arrivals to each queue are neither Possion nor independent,
- Exogeneous outputs are independent and Possion (The state of the entire system is independent of past exogeneous departures.)

9 Application

Routing on the butterfly with Poisson arrivals to the inputs, and exponential transition time through the edges.
Assume that the arrival rate to each of the inputs is λ.
Then the arrival rate to each of the internal nodes is λ.
Let $\mu > \lambda$.
Then the system is stable, expected time in the system is

$$\leq \frac{\log N}{\mu - \lambda}$$