CS 590R–Spring 2003
Algorithms for Communication Networks
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Lecture : #23

Social Networks
Lower Bounds on the Expected Delivery Time

In the last class, we proved that the expected delivery time for the proposed decentralized algorithm when $r = 2$ and $p = q = 1$ is $O(\log^2 n)$. Now, we turn out attention to proving the lower bounds at various values of r, p, and q. A couple of theorems are provided in the subsequent sections that give a polynomial lower bound of the expected delivery time when $r \neq 2$. Consequently, we can conclude that $r = 2$ is the only value for which there is a decentralized algorithm capable of producing routes whose length is a polynomial in $\log n$.

1 Assumptions

Previously, a couple of assumptions have been used when proving the upper bound. These assumptions stated that the message holder u at any step has the knowledge of:

- The set of local contacts of all nodes.
- The location of the target node t.

Here we add one additional assumption:

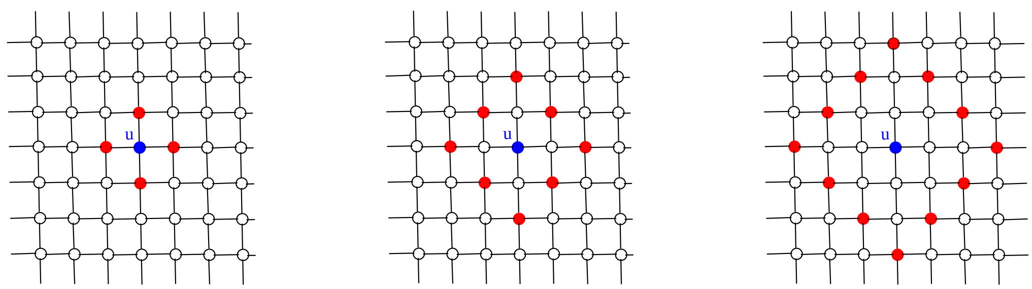
- The current message holder has knowledge of the locations and long-range contacts of all nodes that have come in contact with the message.

In fact this assumption strengthens the obtained lower bound in the sense that the current message holder will not forward the message to any node unless this node has not yet received the message. Thus, steps at which a message revisits a certain node are not counted. Nevertheless, we will show that any decentralized algorithm even with this additional knowledge will have the polynomial lower bound if $r \neq 2$.

2 Terminology

- $\delta = \frac{2-r}{3}$

- U: the set of nodes within lattice distance pn^δ of t (see Fig. 3)
- $\lambda = \left((2-r)2^{7-r}qp^2\right)^{-1}$
- $\mathcal{E}$: the event that the message reaches t within λn^δ steps
- $\mathcal{E}'$: the event that within λn^δ steps, the message reaches a node other than t with a long-range contact in U (see Fig. 4)
- $\mathcal{E}'_i$: the event that $\mathcal{E}'$ happens in step i
- $\mathcal{F}$: the event that the chosen source s and target t are separated by a lattice distance of at least $n/4$
- X: the random variable equal to the number of steps taken for the message to reach t



(a) distance=1 (b) distance=2 (c) distance=3

Figure 1: The number of nodes within a lattice distance of j from a node u is at most $4j$

3 Lower Bound Theorems

Theorem 3.1 *Let $0 \leq r < 2$. There is a constant α_r, depending on p, q, r, but independent of n, so that the expected delivery time of any decentralized algorithm is at least $\alpha_r n^{(2-r)/3}$.*

Proof:

Assume $n \geq n_0$, where n_0 is a sufficiently large fixed constant. We lower bound the expected number of steps required for the message to travel from nodes s to t which are both generated uniformly at random from the grid.

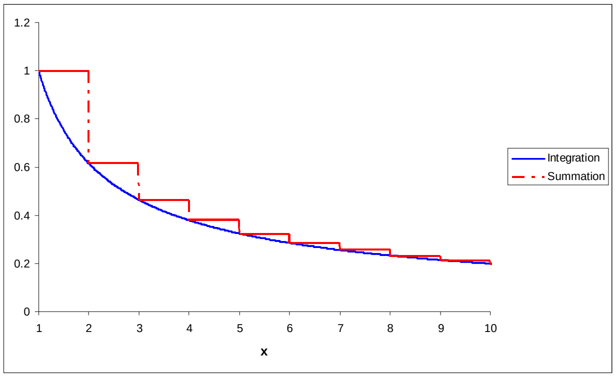


Figure 2: $\sum_{j=1}^{n/2} j^{1-r} \geq \int_1^{n/2} x^{1-r} dx$

The probability that a node u chooses v as its i^{th} out of q long-range contacts is $\frac{d(u,v)^{-r}}{\sum_{u \neq v} d(u,v)^{-r}}$. We have

$$
\begin{aligned}
\sum_{u \neq v} d(u,v)^{-r} &\geq \sum_{j=1}^{n/2} j \cdot j^{-r} \\
&= \sum_{j=1}^{n/2} j^{1-r} \\
&\geq \int_1^{n/2} x^{1-r} dx \quad (\textit{see Fig. } 2) \\
&= \frac{1}{(2-r)} \Big((\frac{n}{2})^{2-r} - 1 \Big) \\
&= \frac{1}{(2-r)} (\frac{n}{2})^{2-r} \Big(1 - (\frac{2}{n})^{2-r} \Big) \\
&\geq \frac{1}{(2-r)2^{3-r}} n^{2-r} \cdot \frac{1}{2} \\
&= \frac{1}{(2-r)2^{3-r}} n^{2-r}
\end{aligned}
$$

Note that: if $n^{2-r} \geq 2^{3-r}$

$$
\begin{aligned}
\Rightarrow \Big(1 - (\frac{2}{n})^{2-r} \Big) &\geq \Big(1 - \frac{2^{2-r}}{2^{3-r}} \Big) \\
&= \Big(1 - \frac{1}{2} \Big) \\
&= \frac{1}{2}
\end{aligned}
$$

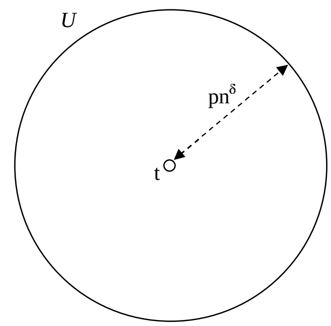


Figure 3: The set U

Now, from the definition of U as shown in Fig. 3, note that $|U| \leq 1 + \sum_{j=1}^{pn^\delta} 4j \leq 4p^2n^{2\delta}$ where we assume n is large enough that $pn^\delta \geq 2$. Moreover, from the definition of $\mathcal{E}'_i$, we have:

$$
\begin{aligned}
\Pr\{\mathcal{E}'_i\} &\leq \frac{q|U|}{\sum_{u \neq v} d(u,v)^{-r}} \\
&\leq \frac{q|U|}{\frac{1}{(2-r)2^{3-r}} n^{2-r}} \\
&\leq \frac{(2-r)2^{3-r} q \cdot 4p^2 n^{2\delta}}{n^{2-r}} \\
&= \frac{(2-r)2^{5-r} qp^2 n^{2\delta}}{n^{2-r}}
\end{aligned}
$$

Thus, from the definition of $\mathcal{E}'$ as shown in Fig. 4 and the assumptions that $\lambda = \left((2-r)2^{7-r}qp^2\right)^{-1}$ and $\delta = (2-r)/3$:

$$
\begin{aligned}
\Pr\{\mathcal{E}'\} &\leq \sum_{i \leq \lambda n^\delta} \Pr(\mathcal{E}'_i) \\
&\leq \lambda n^\delta \cdot \frac{(2-r)2^{5-r} qp^2 n^{2\delta}}{n^{2-r}} \\
&= \frac{1}{(2-r)2^{7-r}qp^2} \cdot \frac{(2-r)2^{5-r} qp^2 n^{3 \cdot \frac{2-r}{3}}}{n^{2-r}} \\
&= 1/4
\end{aligned}
$$

Now with $\mathcal{F}$ denoting the event that the chosen source s and target t are separated by a lattice distance of at least $n/4$. We can show that

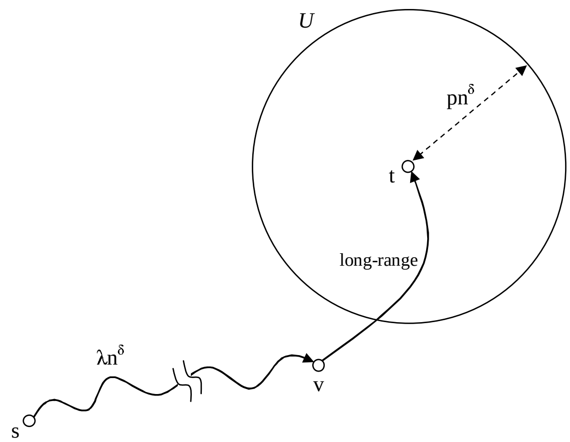


Figure 4: The event $\mathcal{E}'$

$\Pr\{\mathcal{F}\} \geq 1/2$ as follows:

$$\begin{aligned}
\Pr\{\overline{\mathcal{F}}\} &= \Pr\{d(s,t) < \frac{n}{4}\} \\
&= \frac{\textit{number of nodes within a distance} < n/4}{\textit{total number of nodes}} \\
&\leq \frac{\sum_{j=1}^{n/4} 4j}{n^2} \quad (\textit{see Fig. } 1) \\
&= \frac{4(\frac{n}{8})(\frac{n}{4}+1)}{n^2} \\
&= \frac{1}{8} + \frac{1}{2n} \\
&\leq \frac{1}{2} \quad (\textit{since } n \geq 2) \\
\Rightarrow \Pr\{\mathcal{F}\} &= 1 - \Pr\{d(s,t) < \frac{n}{4}\} \\
&\geq \frac{1}{2}
\end{aligned}$$

Now,

$$\begin{aligned}
\Pr\{\mathcal{F} \cap \overline{\mathcal{E}'}\} &= \Pr\{\overline{\overline{\mathcal{F}} \cup \mathcal{E}'}\} \\
&= 1 - \Pr\{\overline{\mathcal{F}} \cup \mathcal{E}'\} \\
&\geq 1 - \Big(\Pr\{\overline{\mathcal{F}}\} + \Pr\{\mathcal{E}'\}\Big) \\
&\geq 1 - (\frac{1}{2} + \frac{1}{4}) \\
&= \frac{1}{4}
\end{aligned}$$

Since X is the random variable denoting the number of steps taken for the message to reach t and $\mathcal{E}$ is the event denoting that the message reaches t within λn^δ steps, and since $(\mathcal{F} \cup \overline{\mathcal{E}'})$ means that nodes are far away and $\mathcal{E}'$ does not happen, we have:

$$\begin{aligned}
& \Pr\{\mathcal{E}|\mathcal{F} \cap \overline{\mathcal{E}'}\} = 0 \\
\Rightarrow \quad & E[X|\mathcal{F} \cap \overline{\mathcal{E}'}] \geq \lambda n^\delta \\
\Rightarrow \quad & E[X] \geq E[X|\mathcal{F} \cap \overline{\mathcal{E}'}] \Pr\{\mathcal{F} \cap \overline{\mathcal{E}'}\} \geq \frac{1}{4}\lambda n^\delta \\
\Rightarrow \quad & E[X] \geq \alpha_r n^{(2-r)/3}
\end{aligned}$$

□

Theorem 3.2 *Let $r > 2$. There is a constant α_r, depending on p, q, r, but independent of n, so that the expected delivery time of any decentralized algorithm is at least $\alpha_r n^{(r-2)/(r-1)}$.*

Proof:

Let $\epsilon = r - 2$. Consider a node u, and let v be a randomly generated long-range contact of u.

$$\begin{aligned}
\Pr\{d(u,v) > m\} &\leq \sum_{j=m+1}^{2n-2} (4j)(j^{-r}) \\
&= 4 \sum_{j=m+1}^{2n-2} j^{1-r} \\
&\leq \int_m^\infty x^{1-r} dx \\
&\leq (r-2)^{-1} m^{2-r} \\
&= \epsilon^{-1} m^{-\epsilon}
\end{aligned}$$

Now, let $\beta = \frac{\epsilon}{1+\epsilon}$, $\gamma = \frac{1}{1+\epsilon}$ and $\lambda' = \frac{min(\epsilon,1)}{8q}$. Assume that $n^\gamma \geq p$.

Moreover, Let $\mathcal{E}'_i$ be the event that in step i, the message reaches a node $u \neq t$ that has a long-range contact v satisfying $d(u,v) > n^\gamma$. Let $\mathcal{E}' = \cup_{i \leq \lambda' n^\beta} \mathcal{E}'_i$ that this happens in the first $\lambda' n^\beta$ steps. We have

$$\begin{aligned}
\Pr(\mathcal{E}') &\leq \sum_{i \leq \lambda' n^\beta} \Pr(\mathcal{E}') \\
&\leq \lambda' n^\beta q \epsilon^{-1} n^{-\epsilon\gamma} \\
&= \lambda' q \epsilon^{-1} \\
&\leq \frac{1}{4}
\end{aligned}$$

The rest of the proof is exactly similar to the previous case. For the complete proof, you may refer to [1]. □

References

[1] J. Kleinberg. *The small-world phenomenon: An algorithmic perspective.* Proc. 32nd ACM Symposium on Theory of Computing, 2000.