Lecture 6

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A Probabilistic Recurrence

We need to show that the expected number of iterations required by randomized MIS is upper-bounded by $O(\log n)$ if the expected reduction in the number of edges at each iteration is atleast greater than a constant fraction of the edges at the beginning of the iteration. To solve for this expectation, let us think of a game: Consider a particle whose position changes in the x direction at discrete time steps and is always at a positive integer. If the particle is currently at position $m > 1$, it proceeds at the next step to position $m - X_m$, where X_m is a random variable over integers $1, \ldots, m-1$. We are only given that $E[X_m] \geq g(m)$, that X_m is chosen independently of the past and g is a monotone non-decreasing function from R^+ to R^+.

Note that in context of the randomized MIS, X_m is a r.v. that denotes the number of edges deleted in an iteration of MIS which started with $|E_m|$ edges.

$$E[X_m] \geq |E_m|/2 = g(|E_m|).$$

So the solution for the expected number of number of iterations for MIS is equivalent to the solution of the following problem. Assuming the particle starts at position n, what is the expected number of steps before it reaches position 1?

Theorem 1. *Let T_n be the random variable denoting the number of steps in which the particle reaches the position 1, starting at n. Then $E[T_n] \leq \int_1^n dx/g(x)$.*

Proof. By induction on n.

Base case: n = 1, is trivially true, since the particle is already at position 1, $E[T_1] = 0$ as required. Suppose the theorem holds for values of m smaller than n. Let $f(m) = \int_1^m dx/g(x)$ for $m \geq 1$.

Consider the first step, during which the particle proceeds from position n to position $n - X_n$, where X_n is chosen from a distribution for which $E[X_n] \geq g(n)$.

we have;

$$E[T_n] = 1 + E[E[T_{n-X_n}]] \leq 1 + E[f(n - X_n)]$$

$$= 1 + E[\int_1^n dy/g(y) - \int_{n-X_n}^n dy/g(y)]$$

using linearity of expectation we have;

$$= 1 + f(n) - E[\int_{n-X_n}^n dy/g(y)]$$

using the fact that g is monotonic increasing we have;
$(g(y) \leq g(n) \cdots y \in [n - X_n, n])$

$$\leq 1 + f(n) - E[\int_{n-X_n}^n dy/g(n)]$$

using $E[X_n] \geq g(n)$ and linearity of expectation have;

$$= 1 + f(n) - E[X_n]/g(n) \leq f(n)$$

□

Remark: Let T_n be a r.v. for number of rounds for MIS . Let $g(n) = n/c$ where $c > 1$ is a constant.

$$E[T_n] \quad \leq \quad \int_1^n dx/g(x)$$

$$
\begin{aligned}
&= c\int_1^n dx/x \\
&= O(\log n)
\end{aligned}
$$

Bounding Deviation from Expectation

Probabilistic inequalities are useful to bound the deviations of a r.v. from its expectation in terms of its moments. The k-th order moment is defined as $E[X^k]$. So in some sense, they estimate the deviations without knowing the underlying distribution of the r.v.

Theorem 2. [Markov's Inequality] *For any non-negative random variable*

$$\Pr(X > a) \leq E[X]/a$$

Proof.

Let Y be a r.v. defined as follows:

$$Y = \begin{cases} 1 & X > a \\ 0 & \textit{otherwise} \end{cases}$$

Therefore by definition of expectation for Y we have;

$$E[Y] = \Pr(X > a)$$

Using the fact that X is non-negative we have;

$$\begin{aligned} Y &<= X/a \\ E[Y] &<= E[X/a] \\ Pr(X > a) &\leq E[X]/a \end{aligned}$$

□

Example: What is the probability of getting more than $3N/4$ heads in N coin flips?

Let Y be the r.v. that counts the number of heads in N coin flips. We know that $E[Y] = N/2$ Using Markov's inequality we have;

$$\Pr(Y > 3N/4) \leq \frac{E[Y]}{3N/4} = \frac{N/2}{3N/4} = 2/3$$

Remark: Markov's inequality gives us a weak bound on the deviation of r.v. Y from its expected value. Since we know that in practice the probability of getting more than $N/2$ heads falls off rapidly assuming binomial distribution Y.

Variance

The variance of a r.v. X is defined as

$$\begin{aligned} Var[X] &= E[(X - E[X])^2] \\ &= E[(X^2 - 2XE[X] + E[X]E[X]) \\ &= E[X^2] - 2E[X]E[X] + E[X]E[X] \\ &= E[X^2] - E^2[X] \end{aligned}$$

The standard deviation of a r.v. X is

$$\sigma(X) = \sqrt{Var[X]}$$

Chebyshev's Inequality

Theorem 3. *For* any *random variable*

$$\Pr(|X - E[X]| \geq a) \leq \frac{Var[X]}{a^2} \quad (1)$$

$$\Pr(|X - E[X]| \geq a\sigma[X]) \leq 1/a^2 \quad (2)$$

$$\Pr(|X - E[X]| \geq \epsilon E[X]) \leq \frac{Var[X]}{\epsilon^2 E^2[X]} \quad (3)$$

Proof. *Using Markov's inequality.*

$$\begin{aligned} \Pr(|X - E[X]| \geq a) &= \Pr((X - E[X])^2 \geq a^2) \\ &\leq \frac{E[(X - E[X])^2]}{a^2} \\ &= \frac{Var[X]}{a^2} \end{aligned}$$

Substituting, a *with* $a\sigma[X] = a\sqrt{Var[X]}$ *in* (1) *we* *have;*

$$\begin{aligned}\Pr(|X - E[X]| \geq a\sigma[X]) &\leq \frac{Var[X]}{(a\sigma[X])^2} \\ &= \frac{Var[X]}{(a\sqrt{Var[X]})^2} \\ &= 1/a^2\end{aligned}$$

Substituting, a *with* $\epsilon E[X]$ *in* (1) *we have;*

$$\Pr(|X - E[X]| \geq \epsilon E[X]) \leq \frac{Var[X]}{\epsilon^2 E^2[X]}$$

□

Remark: Chebyshev's inequality uses second order moments to achieve better deviation bounds than Markov's inequality. It also works for negative r.v. We will see in subsequent lectures, how using higher order moments we can get stronger bounds for the deviations.